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CONDITIONS OF INVERTIBILITY FOR FUNCTIONAL OPERATORS WITH SHIFT IN WEIGHTED HÖLDER SPACES

УМОВИ ОБОРОТНОСТІ ДЛЯ ФУНКЦІОНАЛЬНИХ ОПЕРАТОРІВ ІЗ ЗСУВОМ У ПРОСТОРАХ ГЕЛЬДЕРА З ВАГОЮ

We consider functional operators with shift in weighted Hölder spaces. The main result of this work is the proof of the conditions of invertibility for these operators. We also indicate the forms of the inverse operator. As an application, we propose to use these results for solution of equations with shift which arise in the study of cyclic models for natural systems with renewable resources.

Розглядаються функціональні оператори із зсувом у просторах Гельдера з вагою. Основним результатом роботи є встановлення умов оборотності для цих операторів. Вказані види оберненого оператора. Як застосування запропоновано використовувати отримані результати для розв'язання рівнянь із зсувом, які виникають при досліджені циклічних моделей природних систем з ресурсами, що відновлюються.

1. Introduction. The interest towards the study of functional operators with shift was stipulated by the development of the solvability theory and Fredholm theory for some classes of linear operators, in particular, singular integral operators with Carleman and non-Carleman shift [1 – 3]. Conditions of invertibility for functional operators with shift in weighted Lebesgue spaces were obtained in [1].

Our study of functional operators with shift in the weighted Hölder spaces has an additional motivation: on modeling systems with renewable resources, equations with shift arise in [4, 5], and the theory of linear functional operators with shift is the adequate mathematical instrument for the investigation of such systems.

In Section 2, the boundedness of functional operators with shift in the Hölder spaces and in the weighted Hölder spaces is proved.

In Section 3, some auxiliary lemmas are proved. They will be used in the proof of invertibility conditions.

In Section 4, forms of the inverse operator are given.

In Section 5, conditions of invertibility for functional operators with shift in the Hölder spaces with power wight are obtained. At the end of the article, an application to modeling systems with renewable resources is given.

2. Boundedness of shift operators in the weighted Hölder spaces. We introduce [6] the weighted Hölder spaces $H^0_\mu(J,\rho)$.

A function $\varphi(x)$ that satisfies the following condition on $J=[0,1]$,

$$|\varphi(x_1)-\varphi(x_2)|\leq C|x_1-x_2|^\mu, \qquad x_1\in J, x_2\in J, \quad \mu\in(0,\ 1),$$

is called a Hölder's function with exponent μ and constant C on J.

Let ρ be a power function which has zeros at the endpoints $x=0$, $x=1$:

$$\rho(x)=(x-0)^{\mu_0}(1-x)^{\mu_1}, \qquad \mu<\mu_0<1+\mu, \quad \mu<\mu_1<1+\mu.$$

The functions that become Hölder functions and valued zero at the points $x=0$, $x=1$, after being multiplied by $\rho(x)$, form a Banach space:

$$H^0_\mu(J,\rho), \quad J=[0\,1].$$

The norm in the space $H^0_\mu(J,\rho)$ is defined by

$$\|f(x)\|_{H^0_\mu(J,\rho)} = \|\rho(x)f(x)\|_{H_\mu(J)},$$

where

$$\|\rho(x)f(x)\|_{H_\mu(J)} = \|\rho(x)f(x)\|_C + \|\rho(x)f(x)\|_\mu,$$

and

$$\|\rho(x)f(x)\|_C = \max_{x\in J} |\rho(x)f(x)|,$$

$$\|\rho(x)f(x)\|_\mu = \sup_{x_1,x_2\in J, x_1\neq x_2} |\rho(x)f(x)|_\mu,$$

$$|\rho(x)f(x)|_\mu = \frac{|\rho(x_1)f(x_1)-\rho(x_2)f(x_2)|}{|x_1-x_2|^\mu}.$$

Let $\beta(x)$ be a bijective orientation-preserving shift on J:
if $x_1<x_2$, then $\beta(x_1)<\beta(x_2)$ for any $x_1\in J$, $x_2\in J$; and let $\beta(x)$ have only two fixed points:

$$\beta(0)=0, \qquad \beta(1)=1, \qquad \beta(x)\neq x, \quad \text{when} \quad x\neq 0, \quad x\neq 1.$$

In addition, let $\beta(x)$ be a differentiable function with $\dfrac{d}{dx}\beta(x)\neq 0$ and $\dfrac{d}{dx}\beta(x)\in H_\mu(J)$.

Let us begin with the shift operator $(B_\beta\varphi)(x)=\varphi[\beta(x)]$.

Theorem 1. *Operator B_β is bounded on the space $H_\mu(J)$,*

$$\|B_\beta\|_{\mathcal{B}(H_\mu(J))} \le \|\beta'\|^\mu_C.$$

Operator B_β is bounded on the space $H^0_\mu(J,\rho)$,

$$\|B_\beta\|_{\mathcal{B}(H^0_\mu(J,\rho))} \le \left\|\frac{\rho}{\rho[\beta]}\right\|_{H_\mu(J)} \|B_\beta\|_{\mathcal{B}(H_\mu(J))}.$$

Proof. Let $\varphi\in H_\mu(J)$,

$$\|B_\beta\varphi\|_{H_\mu(J)} = \|B_\beta\varphi\|_C + \|B_\beta\varphi\|_\mu =$$

$$= \|\varphi\|_C + \sup_{x_1\neq x_2} \frac{|\varphi[\beta(x_2)]-\varphi[\beta(x_1)]|\,|\beta(x_2)-\beta(x_1)|^\mu}{|x_2-x_1|^\mu\,|\beta(x_2)-\beta(x_1)|^\mu} \le$$

$$\le \|\varphi\|_C + \sup_{x_1\neq x_2}\left|\frac{\beta(x_2)-\beta(x_1)}{x_2-x_1}\right|^\mu \|\varphi\|_\mu.$$

From here, it follows that

$$\|B_\beta\|_{\mathcal{B}(H_\mu(J))} \le \max\left\{1,\ \sup_{x_1\neq x_2}\left|\frac{\beta(x_2)-\beta(x_1)}{x_2-x_1}\right|^\mu\right\} =$$

$$= \sup_{x_1\neq x_2}\left|\frac{\beta(x_2)-\beta(x_1)}{x_2-x_1}\right|^\mu = \|\beta'\|_C^\mu.$$

Let $\varphi \in H_\mu^0(J,\rho)$; from

$$\left\|\frac{\rho}{\rho[\beta]}B_\beta(\rho\varphi)\right\|_\mu = \sup_{x_1\neq x_2}\left|\frac{\dfrac{\rho(x_1)}{\rho[\beta(x_1)]}\left(B_\beta(\rho\varphi)\right)(x_1) - \dfrac{\rho(x_2)}{\rho[\beta(x_2)]}\left(B_\beta(\rho\varphi)\right)(x_2)}{(x_1-x_2)^\mu}\right| =$$

$$= \sup_{x_1\neq x_2}\left|\frac{\left(\dfrac{\rho(x_1)}{\rho[\beta(x_1)]}-\dfrac{\rho(x_2)}{\rho[\beta(x_2)]}\right)(B_\beta(\rho\varphi))(x_1)+((B_\beta(\rho\varphi))(x_1)-(B_\beta(\rho\varphi))(x_2))\left(\dfrac{\rho(x_2)}{\rho[\beta(x_2)]}\right)}{(x_1-x_2)^\mu}\right| \le$$

$$\le \left\|\frac{\rho}{\rho[\beta]}\right\|_\mu \|B_\beta(\rho\varphi)\|_C + \left\|\frac{\rho}{\rho[\beta]}\right\|_C \left\|B_\beta(\rho\varphi)\right\|_\mu$$

and

$$\left\|\frac{\rho}{\rho[\beta]}\right\|_C \le \left\|\frac{\rho}{\rho[\beta]}\right\|_{H_\mu(J)},$$

it follows that

$$\|B_\beta\varphi\|_{H_\mu^0(J,\rho)} = \|\rho B_\beta\varphi\|_{H_\mu(J)} = \left\|\frac{\rho}{\rho[\beta]}B_\beta(\rho\varphi)\right\|_C + \left\|\frac{\rho}{\rho[\beta]}B_\beta(\rho\varphi)\right\|_\mu \le$$

$$\le \left\|\frac{\rho}{\rho[\beta]}\right\|_C \|B_\beta(\rho\varphi)\|_C + \left\|\frac{\rho}{\rho[\beta]}\right\|_\mu \|B_\beta(\rho\varphi)\|_C + \left\|\frac{\rho}{\rho[\beta]}\right\|_C \|B_\beta(\rho\varphi)\|_\mu \le$$

$$\le \left\|\frac{\rho}{\rho[\beta]}\right\|_{H_\mu(J)} \|B_\beta(\rho\varphi)\|_C + \left\|\frac{\rho}{\rho[\beta]}\right\|_C \|B_\beta(\rho\varphi)\|_\mu \le \left\|\frac{\rho}{\rho[\beta]}\right\|_{H_\mu(J)} \|B_\beta(\rho\varphi)\|_{H_\mu(J)} \le$$

$$\le \left\|\frac{\rho}{\rho[\beta]}\right\|_{H_\mu(J)} \|B_\beta\|_{\mathcal{B}(H_\mu(J))}\|\rho\varphi\|_{H_\mu(J)} = \left\|\frac{\rho}{\rho[\beta]}\right\|_{H_\mu(J)} \|B_\beta\|_{\mathcal{B}(H_\mu(J))}\|\varphi\|_{H_\mu^0(J,\rho)}.$$

Since $\dfrac{\rho(x)}{\rho[\beta(x)]} = \left|\dfrac{x}{\beta(x)}\right|^{\mu_0}\left|\dfrac{1-x}{1-\beta(x)}\right|^{\mu_1} \in H_\mu(J)$, we complete the proof.

Thus the operator $A = aI - bB_\beta$, with coefficients $a \in H_\mu(J)$, $b \in H_\mu(J)$, is bounded on the space $H_\mu^0(J,\rho)$.

3. Auxiliary lemmas. We keep the conditions on the shift β given in Section 2. Without loss of generality, we assume also that for any fixed $x \in (0,1)$,

$$\lim_{m\to+\infty}\beta_m(x) = 0, \qquad \lim_{m\to+\infty}\beta_{-m}(x) = 1;$$

which implies that $\beta^{'}(0) \leq 1$ and $\beta^{'}(1) \geq 1$.

We will use the following notation:

$$r = \mu_0 - \mu, \qquad s = \mu_1 - \mu, \qquad \rho_{r,s}(x) = x^r(1-x)^s, \qquad \rho_{\mu,\mu}(x) = \rho_\mu(x) = x^\mu(1-x)^\mu,$$

$$\rho_{r,s;j}(x) = \rho_{r,s}[\beta_j(x)], \qquad \rho_{\mu;j}(x) = \rho_{\mu,\mu;j}(x), \qquad \beta(x_1, x_2) = \frac{\beta(x_1) - \beta(x_2)}{x_1 - x_2}.$$

Lemma 1. *We have*

$$(\forall \beta(x), x \in J\ )(\forall \varepsilon > 0)(\exists n_0 \in \mathbf{N})(\forall x \in J)(\exists n_1, n_2 \in \mathbf{N}, n_1 < n_2, n_0 = n_2 - n_1)$$

$$\Big[\beta_n(x) \in [0, \varepsilon] \bigcup [1-\varepsilon, 1],\ n \in \mathbf{N} \setminus [n_1, n_2]\Big].$$

An essential point here is that $n_0 = n_2 - n_1$ is independent of x.

Proof. Follows directly from the properties of $\beta(x)$.

Lemma 2. *Under the conditions*

$$a(x) \neq 0; \qquad \left|\beta'(0)\right|^{-\mu_0+\mu} \left|\frac{b(0)}{a(0)}\right| < 1, \qquad \left|\beta'(1)\right|^{-\mu_1+\mu} \left|\frac{b(1)}{a(1)}\right| < 1, \tag{1}$$

the following inequalities hold in some one-sided ε_1-neighborhoods of the endpoints $x = 0$, $x = 1$:

$$\left|u(x)\frac{\rho_{r,s}(x)}{\rho_{r,s;1}(x)}\right| \leq q_1 < 1, \qquad x \in [0, \varepsilon_1] \bigcup [1 - \varepsilon_1, 1]. \tag{2}$$

Proof. Follows from (1) and from the properties of $\beta(x)$, $a(x)$, $b(x)$.

Lemma 3. *Under the condition* (1), *there is $\varepsilon_2 > 0$ such that the following inequality holds*:

$$\left|\left|\frac{\beta(x_1) - \beta(x_2)}{x_1 - x_2}\right|^\mu u(x_2)\frac{\rho(x_2)}{\rho[\beta(x_2)]}\right| \leq q_2 < 1, \tag{3}$$

for $x_1, x_2 \in [0, \varepsilon_2]$ or $x_1, x_2 \in [1 - \varepsilon_2, 1]$, or $x_1 \in [0, \varepsilon_2]$, $x_2 \in [1 - \varepsilon_2, 1]$.

Proof. It is easy to see that the following identity:

$$\left|\left|\frac{\beta(x_1) - \beta(x_2)}{x_1 - x_2}\right|^\mu u(x_2)\frac{\rho(x_2)}{\rho[\beta(x_2)]}\right| =$$

$$= \left|\frac{\beta(x_1) - \beta(x_2)}{x_1 - x_2}\right|^\mu \left|\frac{x_2}{\beta(x_2)}\right|^\mu \left|\frac{1 - x_2}{1 - \beta(x_2)}\right|^\mu \left|u(x_2)\frac{\rho_{r,s}(x_2)}{\rho_{r,s;1}(x_2)}\right| \tag{4}$$

holds. We estimate then the expression $\left|\dfrac{\beta(x_1) - \beta(x_2)}{x_1 - x_2}\right|^\mu \left|\dfrac{x_2}{\beta(x_2)}\right|^\mu \left|\dfrac{1 - x_2}{1 - \beta(x_2)}\right|^\mu$.

By (2) of Lemma 2 and $\lim_{x_1, x_2 \to 0} \left|\dfrac{\beta(x_1) - \beta(x_2)}{x_1 - x_2}\right|^\mu \left|\dfrac{x_2}{\beta(x_2)}\right|^\mu = 1$, $\lim_{x_2 \to 0} \left|\dfrac{x_2 - 1}{\beta(x_2) - 1}\right|^\mu = 1$, we can choose $\varepsilon_3 > 0$ such that the inequality $\left|\dfrac{\beta(x_1) - \beta(x_2)}{x_1 - x_2}\right|^\mu \left|\dfrac{x_2}{\beta(x_2)}\right|^\mu \left|\dfrac{1 - x_2}{1 - \beta(x_2)}\right|^\mu q_1 \leq q_3 < 1$ holds for $x_1, x_2 \in [0, \varepsilon_3]$.

By (2) of Lemma 2 and $\lim_{x_1,x_2\to 1}\left|\dfrac{\beta(x_1)-\beta(x_2)}{x_1-x_2}\right|^{\mu}\left|\dfrac{1-x_2}{1-\beta(x_2)}\right|^{\mu}=1$, $\lim_{x_2\to 1}\left|\dfrac{x_2}{\beta(x_2)}\right|^{\mu}=1$, we can choose $\varepsilon_4>0$ such that the inequality $\left|\dfrac{\beta(x_1)-\beta(x_2)}{x_1-x_2}\right|^{\mu}\left|\dfrac{x_2}{\beta(x_2)}\right|^{\mu}\left|\dfrac{1-x_2}{1-\beta(x_2)}\right|^{\mu}q_1\le q_4<1$ holds for $x_1,x_2\in[1-\varepsilon_4,1]$.

As

$$\lim_{x_1\to 0,x_2\to 1}\left|\frac{\beta(x_1)-\beta(x_2)}{x_1-x_2}\right|^{\mu}=1,\qquad \lim_{x_2\to 1}\left|\frac{x_2}{\beta(x_2)}\right|^{\mu}=1,$$

$$\lim_{x_2\to 1}\left|\frac{1-x_2}{1-\beta(x_2)}\right|^{\mu}=\left|\beta'(1)\right|^{-\mu}\le 1$$

we can choose $\varepsilon_5>0$ such that the inequality

$$\left|\frac{\beta(x_1)-\beta(x_2)}{x_1-x_2}\right|^{\mu}\left|\frac{x_2}{\beta(x_2)}\right|^{\mu}\left|\frac{1-x_2}{1-\beta(x_2)}\right|^{\mu}q_1\le q_5<1$$

holds for $x_1\in[0,\varepsilon_5]$, $x_2\in[1-\varepsilon_5,1]$.

To prove inequality (3), it is sufficient to choose $\varepsilon_2=\min(\varepsilon_3,\varepsilon_4,\varepsilon_5)$, take $q_2=\max(q_3,q_4,q_5)$ and apply the obtained estimates to expression (4).

By Lemma 1, for $\varepsilon=\min(\varepsilon_1,\varepsilon_2)$ there exists a positive integer n_0 such that for each $x\in[0,1]$ at most n_0 values of $\beta_n(x)$ is outside of $[0,\varepsilon]\bigcup[1-\varepsilon,1]$. Let $q=\max(q_1,q_2)$.

Lemma 4. *If $\{w(x)\}|_{x=0}=0$, then $\|w(x)\|_{H_\mu(J)}\ge\sup_{0<x<1}\dfrac{|w(x)|}{x^{\mu}}$.*

If $\{w(x)\}|_{x=1}=0$, then $\|w(x)\|_{H_\mu(J)}\ge\sup_{0<x<1}\dfrac{|w(x)|}{(1-x)^{\mu}}$.

If $\{w(x)\}|_{x=0}=\{w(x)\}|_{x=1}=0$, then $\|w(x)\|_{H_\mu(J)}\ge\left(\dfrac{1}{2}\right)^{\mu}\sup_{0<x<1}\dfrac{|w(x)|}{x^{\mu}(1-x)^{\mu}}$.

For $\varphi\in H^0_\mu(J,\rho)$, the inequality

$$\left\|\frac{\rho\varphi}{\rho_{\mu,\mu}}\right\|_C\le 2^{\mu}\,\|\varphi\|_{H^0_\mu(J,\rho)} \tag{5}$$

holds.

Proof. The proof follows from the inequalities

$$\|w(x)\|_{C(J)}+\sup_{\substack{x_1,x_2\in J\\ x_1\ne x_2}}\frac{|w(x_1)-w(x_2)|}{|x_1-x_2|^{\mu}}\ge$$

$$\ge\|w(x)\|_{C(J)}+\sup_{0<x_1<1}\frac{|w(x_1)-0|}{|x_1-0|^{\mu}}\ge\sup_{0<x<1}\frac{|w(x)|}{x^{\mu}},$$

$$\|w(x)\|_{C(J)}+\sup_{\substack{x_1,x_2\in J\\ x_1\ne x_2}}\frac{|w(x_1)-w(x_2)|}{|x_1-x_2|^{\mu}}\ge$$

$$\ge\|w(x)\|C(J)+\sup_{0<x_2<1}\frac{|0-w(x_2)|}{|1-x_2|^{\mu}}\ge\sup_{0<x<1}\frac{|w(x)|}{(1-x)^{\mu}},$$

$$\|w(x)\|_{H_\mu(J)} = \frac{1}{2}\big(\|w(x)\|_{H_\mu(J)} + \|w(x)\|_{H_\mu(J)}\big) \geq$$

$$\geq \frac{1}{2}\left(\sup_{0<x<1}\frac{|w(x)|}{x^\mu} + \sup_{0<x<1}\frac{|w(x)|}{(1-x)^\mu}\right) \geq$$

$$\geq \frac{1}{2}\sup_{0<x<1}\frac{\left((1-x)^\mu + x^\mu\right)|w(x)|}{x^\mu(1-x)^\mu} \geq \left(\frac{1}{2}\right)^\mu \sup_{0<x<1}\frac{|w(x)|}{x^\mu(1-x)^\mu}.$$

It remains to prove (5):

$$\left\|\frac{\rho\varphi}{\rho_{\mu,\mu}}\right\|_C \leq 2^{\mu-1}\left\|\frac{x^\mu+(1-x)^\mu}{x^\mu(1-x)^\mu}\rho(x)\varphi(x)\right\|_C \leq$$

$$\leq 2^{\mu-1}\left[\left\|\frac{\rho(x)\varphi(x)}{(1-x)^\mu}\right\|_C + \left\|\frac{\rho(x)\varphi(x)}{(x)^\mu}\right\|_C\right] \leq$$

$$\leq 2^{\mu-1}\left[\left\|\frac{\rho(1)\varphi(1)-\rho(x_2)\varphi(x_2)}{(1-x_2)^\mu}\right\|_C + \left\|\frac{\rho(x_1)\varphi(x_1)-\rho(0)\varphi(0)}{(x_1-0)^\mu}\right\|_C\right] \leq$$

$$\leq 2^{\mu-1}\left[\left\|\frac{\rho(x_1)\varphi(x_1)-\rho(x_2)\varphi(x_2)}{(x_1-x_2)^\mu}\right\|_C + \left\|\frac{\rho(x_1)\varphi(x_1)-\rho(x_2)\varphi(x_2)}{(x_1-x_2)^\mu}\right\|_C\right] =$$

$$= 2^\mu\|\rho\varphi\|_\mu \leq 2^\mu\|\rho\varphi\|_{H_\mu(J)} = 2^\mu\|\varphi\|_{H^0_\mu(J,\rho)}.$$

In the above, we use that $1 \leq 2^{\mu-1}\big|x^\mu + (1-x)^\mu\big|$, $\rho(0)\varphi(0) = \rho(1)\varphi(1) = 0$.

We will use these lemmas in the proof of invertibility conditions in Section 5.

4. Structure of the inverse operator. The operators

$$A = aI - bB_\beta,$$

where $a \in H_\mu$, $b \in H_\mu$, $a \neq 0$, and

$$U = I - uB_\beta,$$

where $u = b/a$, are invertible simultaneously on the weighted Hölder space $H^0_\mu(J,\rho)$.

If there exists a natural number n such that

$$\left\|\left(\prod_{j=0}^{n-1}u_j(x)\right)B^n_\beta\right\|_{\mathcal{B}(H^0_\mu(J,\rho))} < 1,$$

where

$$u_j(x) = u\big[\beta_j(x)\big],$$

then the operator U is invertible on $H^0_\mu(J,\rho)$ and

$$U^{-1} = \left(I + uB_\beta + \ldots + \left(\prod_{j=0}^{n-2}u_j(x)\right)B^{n-1}_\beta\right)\left(I - \left(\prod_{j=0}^{n-1}u_j(x)\right)B^n_\beta\right)^{-1}.$$

This statement was proved in [1] for weighted Lebesgue spaces. The proof for the weighted Hölder spaces literally follows the above one as an application of algebraic operations does not depend on the specific properties of the spaces.

We note that the inverse operator U^{-1} can be written in the form

$$U^{-1} = \left(I + uB_\beta + \ldots + \left(\prod_{j=0}^{m-2} u_j(x) \right) B_\beta^{m-1} \right) \left(I - \left(\prod_{j=0}^{m-1} u_j(x) \right) B_\beta^m \right)^{-1},$$

with any another $m, \ m \neq n$, subject to the condition $\left\| \prod_{j=0}^{m-1} u_j(x) B_\beta^m \right\|_{\mathcal{B}(H_\mu^0(J,\rho))} < 1$.

Analogously, if $b \neq 0$ and there exists a natural number n such that

$$\left\| \left(\prod_{j=0}^{n-1} v_j(x) \right) B_\beta^{-n} \right\|_{\mathcal{B}(H_\mu^0(J,\rho))} < 1,$$

where

$$v(x) = \frac{a\left[\beta^{-1}(x)\right]}{b\left[\beta^{-1}(x)\right]}, \qquad v_j(x) = v\left[\beta_j^{-1}(x)\right],$$

then the operator

$$V = I - vB_\beta^{-1}$$

is invertible on the space $\mathcal{B}(H_\mu^0(J,\rho))$ and its inverse operator is given by

$$V^{-1} = \left(I + vB_\beta^{-1} + \ldots + \left(\prod_{j=0}^{n-2} v_j(x) \right) B_\beta^{-n+1} \right) \left(I - \left(\prod_{j=0}^{n-1} v_j(x) \right) B_\beta^{-n} \right)^{-1}.$$

It is obvious that $A = -bB_\beta \left[I - \left(B_\beta^{-1} \frac{a}{b} \right) B_\beta^{-1} \right]$, $A^{-1} = -V^{-1} B_\beta^{-1} \left(\frac{1}{b} \right) I$.

5. Invertibility conditions for the operator A on the weighted Hölder spaces. We will use the following notation

$$f_n(x) = \left(B_\beta^n f \right)(x), \qquad \beta_n^\mu(x_1, x_2) = \left| \frac{\beta_{n+1}(x_1) - \beta_{n+1}(x_2)}{\beta_n(x_1) - \beta_n(x_2)} \right|^\mu, \qquad \tilde{u}_j(x) = \frac{\rho_j(x)}{\rho_{j+1}(x)} u_j(x),$$

Theorem 2. *Conditions* (1) *implies that there exists a natural number* n *for which*

$$\left\| \left(\prod_{j=0}^{n-1} u_j \right) B_\beta^n \right\|_{H_\mu^0(J,\rho)} < 1.$$

Proof. To prove that

$$\left\| \rho \left(\prod_{j=0}^{n-1} u_j \right) B_\beta^n \varphi \right\|_C + \left\| \rho \left(\prod_{j=0}^{n-1} u_j \right) B_\beta^n \varphi \right\|_\mu < \|\varphi\|_{H_\mu^0(J,\rho)}, \tag{6}$$

we estimate each summand separately. For the first one we have

$$\left\|\rho\left(\prod_{j=0}^{n-1}u_j\right)B_\beta^n\varphi\right\|_C=\left\|\rho\frac{\rho_{r,s}}{\rho_{r,s}}\left(\prod_{j=0}^{n-1}u_j\right)\frac{1}{\rho_{r,s;n}}B_\beta^n(\rho_{r,s}\varphi)\right\|_C\le$$

$$\le\left\|\rho_{\mu,\mu}\prod_{j=0}^{n-1}u_j\frac{\rho_{r,s;j}}{\rho_{r,s;j+1}}\right\|_C\|\rho_{r,s}\varphi\|_C\le\left\|\rho_{\mu,\mu}\prod_{j=0}^{n-1}u_j\frac{\rho_{r,s;j}}{\rho_{r,s;j+1}}\right\|_C 2^\mu\|\varphi\|_{H_\mu^0(J,\rho)}\,. \tag{7}$$

We took into account

$$\frac{\rho}{\rho_{r,s}}=\rho_{\mu,\mu},\qquad\frac{\rho_{r,s}}{\rho_{r,s;n}}=\prod_{j=0}^{n-1}\frac{\rho_{r,s;j}}{\rho_{r,s;j+1}},\qquad\left\|B_\beta^n(\rho_{r,s}\varphi)\right\|_C=\|(\rho_{r,s}\varphi)\|_C$$

and inequality (5) from Lemma 4.

By (2) of Lemma 2, it follows that the first factor on the right-hand side of inequality (7) $\left\|\rho_{\mu,\mu}\prod_{j=0}^{n-1}u_j\frac{\rho_{r,s;j}}{\rho_{r,s;j+1}}\right\|_C$ tends to zero when $n\to\infty$.

Now, we estimate the second summand of (6). We use the following notation:

$$f_n(x)=\left(B_\beta^n f\right)(x),\qquad\beta_n^\mu(x_1,x_2)=\left|\frac{\beta_{n+1}(x_1)-\beta_{n+1}(x_2)}{\beta_n(x_1)-\beta_n(x_2)}\right|^\mu,\qquad\tilde u_j(x)=\frac{\rho_j(x)}{\rho_{j+1}(x)}u_j(x).$$

We have

$$\left\|\rho\left(\prod_{j=0}^{n-1}u_j\right)B_\beta^n\varphi\right\|_\mu=\left\|\prod_{j=0}^{n-1}\tilde u_j\rho_n\varphi_n\right\|_\mu\le$$

$$\le\sup_{x_1<x_2}\frac{\left|\prod_{j=0}^{n-1}\tilde u_j(x_1)\rho_n(x_1)\varphi_n(x_1)-\prod_{j=0}^{n-1}\tilde u_j(x_2)\rho_n(x_2)\varphi_n(x_2)\right|}{|x_1-x_2|^\mu}=$$

$$=\sup_{x_1<x_2}\frac{\left|(\rho_n\varphi_n)(x_1)\Big(\prod_{j=0}^{n-1}\tilde u_j(x_1)-\prod_{j=0}^{n-1}\tilde u_j(x_2)\Big)+\prod_{j=0}^{n-1}\tilde u_j(x_2)\left((\rho_n\varphi_n)(x_1)-(\rho_n\varphi_n)(x_2)\right)\right|}{|x_1-x_2|^\mu}\le$$

$$\le\sup_{x_1<x_2}\frac{\left|(\rho_n\varphi_n)(x_1)\left(\tilde u(x_1)-\tilde u(x_2)\right)\prod_{j=0}^{n-2}\tilde u_{j+1}(x_1)+\left(\tilde u_{n-1}(x_1)-\tilde u_{n-1}(x_2)\right)\prod_{j=0}^{n-2}\tilde u_j(x_2)\right|}{|x_1-x_2|^\mu}+$$

$$+\sup_{x_1<x_2}\frac{\left|(\rho_n\varphi_n)(x_1)\sum_{j=0}^{n-3}\Big(\left(\tilde u_{j+1}(x_1)-\tilde u_{j+1}(x_2)\right)\prod_{i=j}^{n-3}\tilde u_{i+2}(x_1)\prod_{k=0}^{j}\tilde u_k(x_2)\Big)\right|}{|x_1-x_2|^\mu}+$$

$$+\sup_{x_1<x_2}\frac{|(\rho_n\varphi_n)(x_1)-(\rho_n\varphi_n)(x_2)|}{|\beta_n(x_1)-\beta_n(x_2)|^\mu}\sup_{x_1<x_2}\left|\prod_{j=0}^{n-1}\tilde u_j(x_2)\frac{|\beta_n(x_1)-\beta_n(x_2)|^\mu}{|x_1-x_2|^\mu}\right|.$$

Here, we used the identities

$$\prod_{j=0}^{n-1}\tilde{u}_j(x_1)-\prod_{j=0}^{n-1}\tilde{u}_j(x_2)=$$

$$=(\tilde{u}(x_1)-\tilde{u}(x_2))\prod_{j=0}^{n-2}\tilde{u}_{j+1}(x_1)+(\tilde{u}_{n-1}(x_1)-\tilde{u}_{n-1}(x_2))\prod_{j=0}^{n-2}\tilde{u}_j(x_2)+$$

$$+\tilde{u}(x_2)\prod_{j=0}^{n-2}\tilde{u}_{j+1}(x_1)-\tilde{u}_{n-1}(x_1)\prod_{j=0}^{n-2}\tilde{u}_j(x_2)$$

and

$$\tilde{u}(x_2)\prod_{j=0}^{n-2}\tilde{u}_{j+1}(x_1)-\tilde{u}_{n-1}(x_1))\prod_{j=0}^{n-2}\tilde{u}_j(x_2)=$$

$$=\sum_{j=0}^{n-3}\left((\tilde{u}_{j+1}(x_1)-\tilde{u}_{j+1}(x_2))\prod_{i=j}^{n-3}\tilde{u}_{i+2}(x_1)\prod_{k=0}^{j}\tilde{u}_k(x_2)\right).$$

Finally, taking into account the identities

$$\frac{\rho_{\mu;n}(x_1)}{\rho_{\mu;j+2}(x_1)}=\prod_{i=j}^{n-3}\frac{\rho_{\mu;i+3}(x_1)}{\rho_{\mu;i+2}(x_1)},\qquad \left|\frac{\beta_{j+1}(x_1)-\beta_{j+1}(x_2)}{x_1-x_2}\right|^{\mu}=\prod_{k=0}^{j}\beta_k^{\mu}(x_1,x_2),$$

we get

$$\left\|\rho\left(\prod_{j=0}^{n-1}u_j\right)B_\beta^n\varphi\right\|_\mu\le\|\tilde{u}\|_\mu\sup_{x_1<x_2}\left|\frac{\rho_n(x_1)\varphi_n(x_1)}{\rho_{\mu;n}(x_1)}\rho_{\mu;1}(x_1)\prod_{j=0}^{n-2}\tilde{u}_{j+1}(x_1)\frac{\rho_{\mu;n}(x_1)}{\rho_{\mu;1}(x_1)}\right|+$$

$$+\sup_{x_1<x_2}\left|\rho_n(x_1)\varphi_n(x_1)\frac{(\tilde{u}_{n-1}(x_1)-\tilde{u}_{n-1}(x_2))}{(\beta_{n-1}(x_1)-\beta_{n-1}(x_2))^{\mu}}\prod_{j=0}^{n-2}\tilde{u}_j(x_2)\frac{(\beta_{n-1}(x_1)-\beta_{n-1}(x_2))^{\mu}}{|x_1-x_2|^{\mu}}\right|+$$

$$+\sup_{x_1<x_2}\left|\frac{\rho_n(x_1)\varphi_n(x_1)}{\rho_{\mu;n}(x_1)}\sum_{j=0}^{n-3}\left(\frac{(\tilde{u}_{j+1}(x_1)-\tilde{u}_{j+1}(x_2))}{(\beta_{j+1}(x_1)-\beta_{j+1}(x_2))^{\mu}}\frac{(\beta_{n-1}(x_1)-\beta_{n-1}(x_2))^{\mu}}{|x_1-x_2|^{\mu}}\times\right.\right.$$

$$\left.\left.\times\rho_{\mu;j+2}(x_1)\times\prod_{i=j}^{n-3}\tilde{u}_{i+2}(x_1)\frac{\rho_{\mu;n}(x_1)}{\rho_{\mu;j+2}(x_1)}\prod_{k=0}^{j}\tilde{u}_k(x_2)\right)\right|+$$

$$+\left\|\frac{(\rho\varphi)(x_1)-(\rho\varphi)(x_2))}{(\beta_n(x_1)-\beta_n(x_2))^{\mu}}\right\|_C\sup_{x_1<x_2}\left|\prod_{j=0}^{n-1}\tilde{u}_j(x_2)\frac{(\beta_n(x_1)-\beta_n(x_2))^{\mu}}{(x_1-x_2)^{\mu}}\right|\le$$

$$\leq \|\tilde{u}\|_{\mu} \left\| \frac{\rho_n \varphi_n}{\rho_{\mu;n}} \right\|_C \sup_{x_1<x_2} \left| \rho_{\mu;1}(x_1) \prod_{j=0}^{n-2} \tilde{u}_{j+1}(x_1) \frac{\rho_{\mu;j+2}(x_1)}{\rho_{\mu;j+1}(x_1)} \right| +$$

$$+ \|\tilde{u}\|_{\mu} \|\rho_n \varphi_n\|_C \sup_{x_1<x_2} \left| \prod_{j=0}^{n-2} \tilde{u}_j(x_2) \beta_j^{\mu}(x_1, x_2) \right| +$$

$$+ \|\tilde{u}\|_{\mu} \left\| \frac{\rho_n \varphi_n}{\rho_{\mu;n}} \right\|_C \sup_{x_1<x_2} \left| \sum_{j=0}^{n-3} \rho_{\mu;j+2} \left(\prod_{i=j}^{n-3} \tilde{u}_{i+2}(x_1) \frac{\rho_{\mu;i+3}(x_1)}{\rho_{\mu;i+2}(x_1)} \prod_{k=0}^{j} \tilde{u}_k(x_2) \beta_k^{\mu}(x_1, x_2) \right) \right| +$$

$$+ \|\rho\varphi\|_{\mu} \sup_{x_1<x_2} \left| \prod_{j=0}^{n-1} \tilde{u}_j(x_2) \beta_j^{\mu}(x_1, x_2) \right|.$$

By (2) of Lemma 2, the inequality

$$\left| \tilde{u}_{l+1}(x_1) \frac{\rho_{\mu;l}(x_1)}{\rho_{\mu;l+1}(x_1)} \right| \leq q < 1 \tag{8}$$

holds for every fixed x_1 with a possible exception of n_0 values of l.

From Lemma 1 it follows that only n_0 values of $\beta_l(x_1)$ may be outside of the set $[0, \varepsilon] \bigcup [1-\varepsilon, 1]$, where inequality (8) holds. Here the number n_0 is from Lemma 1.

By (3) of Lemma 3, the inequality

$$\left| \tilde{u}_l(x_2) \beta_l^{\mu}(x_1, x_2) \right| \leq q < 1 \tag{9}$$

holds for all fixed $x_1, x_2,\ x_1 < x_2$ with a possible exception of $2n_0$ values of l. In fact, under $x_1 < x_2$, we have only two failures of the condition $\beta_l(x_1), \beta_l(x_2) \in [0, \varepsilon] \bigcup [1-\varepsilon, 1]$, and $\beta_l(x_1) \in$ $\in [0, \varepsilon]$, $\beta_l(x_2) \in [1-\varepsilon, 1]$, where inequality (9) holds. It means that the failures may occur when $\beta_l(x_1) \in (\varepsilon, 1-\varepsilon)$, or $\beta_l(x_2) \in (\varepsilon, 1-\varepsilon)$. According to Lemma 1, there are no more than n_0 values of $\beta_l(x_1)$ in $(\varepsilon, 1-\varepsilon)$ and there are no more than n_0 values of $\beta_l(x_2)$ in $(\varepsilon, 1-\varepsilon)$.

We have

$$\left\| \rho \left(\prod_{j=0}^{n-1} u_j \right) B_{\beta}^n \varphi \right\|_{\mu} \leq \Bigg\{ \|\tilde{u}\|_{\mu} 2^{\mu} \|\rho_{\mu,\mu}\|_C q^{n-1-m_0} M^{m_0} + \|\tilde{u}\|_{\mu} q^{n-1-2m_0} M^{2m_0} +$$

$$+ \|\tilde{u}\|_{\mu} 2^{\mu} \|\rho_{\mu,\mu}\|_C \sum_{j=0}^{n-3} q^{n-2-j-m_0} M^{m_0} q^{j+1-2m_0} M^{2m_0} + q^{n-2m_0} M^{2m_0} \Bigg\} \|\varphi\|_{H_{\mu}^0(J,\rho)}, \tag{10}$$

where the constant M is given by

$$M = \max \left(\left\| \tilde{u}(x) \frac{\rho_{\mu;1}(x)}{\rho_{\mu\mu}(x)} \right\|_C, \|\tilde{u}(x_2) \beta^{\mu}(x_1, x_2)\|_C \right).$$

Here, inequalities (5), (8) and (9) were used. The factor in the brackets of (10) tends to zero when $n \to \infty$.

Thus, there exists n such that

$$\left\| \left(\prod_{j=0}^{n-1} u_j \right) B_\beta^n \varphi \right\|_{H_\mu^0(J,\rho)} < \|\varphi\|_{H_\mu^0(J,\rho)} ,$$

which means that the operator $U = I - uB_\beta$ is invertible on the space $H_\mu^0(J,\rho)$.

Theorem 2 is proved.

Theorem 3. *The operator A acting on Banach space $H_\mu^0(J,\rho)$, is invertible if the following condition holds*:

$$\sigma_\beta[a(x), b(x)] \neq 0, \quad x \in J,$$

where the function σ_β is defined by

$$\sigma_\beta[a(x), b(x)] = \begin{cases} a(x), & \textit{when} \quad |a(i)| > [\beta'(i)]^{-\mu_i+\mu} \, |b(i)|, \quad i = 0, 1, \\ b(x), & \textit{when} \quad |a(i)| < [\beta'(i)]^{-\mu_i+\mu} \, \big|b(i)\big|, \quad i = 0, 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We consider only the case

$$\begin{gathered} a(x) \neq 0, \quad x \in J, \\ |a(i)| > |\beta'(i)|^{-\mu_i+\mu} |b(i)|, \quad i = 0, 1. \end{gathered} \tag{11}$$

The case

$$\begin{gathered} b(x) \neq 0, \quad x \in J, \\ |a(i)| < |\beta'(i)|^{-\mu_i+\mu} |b(i)|, \quad i = 0, 1, \end{gathered}$$

can be considered analogously.

Recall that the operators $aI - bB_\beta$ and $U = I - uB_\beta$, where $u = b/a$, are invertible simultaneously on $H_\mu^0(J,\rho)$.

Thus, there exists n such that

$$\left\| \left(\prod_{j=0}^{n-1} u_j \right) B_\beta^n \varphi \right\|_{H_\mu^0(J,\rho)} < ||\varphi||_{H_\mu^0(J,\rho)} ,$$

which means that operator $U = I - uB_\beta$ is invertible in space $H_\mu^0(J,\rho)$.

Theorem 3 is proved.

Now, we will focus on the application of the above results to a modeling of systems with renewable resources. For the study of such systems, cyclic models based on functional operators with shift were elaborated in [4]. The Balance relation describing the state of cyclic equilibrium is the equation $aI\nu - bB_\beta\nu = g$ for the unknown distribution function $\nu \in H_\mu^0(J,\rho)$.

In [5], a reproductive summand has been added for a more accurate description of the process of reproduction; this term has been expressed by integrals with degenerate kernels.

If we model the behavior of a system with two resources, taking into account the interaction between them, by integrals with degenerate kernels and follow the principles of modeling from [4], we will obtain two equations with two unknowns, ν_1 and ν_2:

$$a_1(x)\nu_1(x) - b_1(x)\nu_1[\beta_1(x)] + \Sigma_1(x) + \Gamma_1(x) = g_1(x), \tag{12}$$

$$a_2(x)\nu_2(x) - b_2(x)\nu_2[\beta_2(x)] + \Sigma_2(x) + \Gamma_2(x) = g_2(x), \tag{13}$$

where ν_1 and ν_2 are the densities of the distributions of the first and second resources by their respective individual parameters (such as weight or length), and

$$\Sigma_1(x) = \sum_{i=1}^{m_1} \int_J \zeta_{1,i}(x)\xi_{1,i}(t)\nu_1(t)dt, \qquad \Gamma_1(x) = \sum_{i=1}^{n_1} \int_J \varrho_{1,i}(x)\delta_{1,i}(t)\nu_2(t)dt,$$

$$\Sigma_2(x) = \sum_{i=1}^{m_2} \int_J \zeta_{2,i}(x)\xi_{2,i}(t)\nu_2(t)dt, \qquad \Gamma_2(x) = \sum_{i=1}^{n_2} \int_J \varrho_{2,i}(x)\delta_{2,i}(t)\nu_1(t)dt,$$

are the terms of reproduction and interaction process respectively.

We consider our model on the space $H^0_\mu(J,\rho)$ Suppose that for

$$A_1 = a_1(x)\nu_1(x) - b_1(x)\nu_1[\beta_1(x)], \qquad A_2 = a_2(x)\nu_2(x) - b_2(x)\nu_2[\beta_2(x)]$$

on $H^0_\mu(J,\rho)$ the invertibility conditions of Theorem 3 hold. Thus, the inverse operators A_1^{-1} and A_2^{-1} for A_1 and A_2 exist. We apply these inverse operators to the left-hand side of equations (12), (13) and obtain Fredholm equations of the second type with degenerate kernels. Using a known method of solving such equations, we can find densities ν_1 and ν_2 of the cyclic equilibrium of the system.

1. *Karlovich Yu. I., Kravchenko V. G.* Singular integral equations with non-Carleman shift on an open contour // Different. Equat. – 1981. – **17**, № 2. – P. 2212 – 2223.
2. *Litvinchuk G. S.* Solvability theory of boundary value problems and singular integral equations with shift. – Dordrecht etc.: Kluwer Acad. Publ., 2000. – 378 p.
3. *Kravchenko V. G., Litvinchuk G. S.* Introduction to the theory of singular integral operators with shift. – Dordrecht etc.: Kluwer Acad. Publ., 1994. – 288 p.
4. *Tarasenko A., Karelin A., Lechuga G. P., Hernández M. G.* Modelling systems with renewable resources based on functional operators with shift // Appl. Math. and Comput. – 2010. – **216**, № 7. – P. 1938 – 1944.
5. *Karelin O., Tarasenko A., Hernández M. G.* Application of functional operators with shift to the study of renewable systems when the reproductive processed is describedby integrals with degenerate kernels // Appl. Math. – 2013. – **4**. – P. 1376 – 1380.
6. *Duduchava R. V.* Unidimensional singular integral operator algebras in spaces of Holder functions with weight (in Russian) // Trudy Tbilisskogo Mat. Inst. Acad. Nauk Gruz.SSR. – 1973. – **43**. – P. 19 – 52.

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