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Received 12 December 2005, revised 10 September 2006, accepted 27 December 2006
Published online 6 June 2007

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**MSC (2000)** Primary: 47G10, 45E99, 45F15

*Dedicated to Professor Frank-Olme Speck on the occasion of his 60th birthday*

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Math. Nachr. **280**, No. 9–10, 1108–1117 (2007) / **DOI** 10.1002/mana.200510539

WILEY-VCH

Math. Nachr. **280,** No. 9–10, 1108 – 1117 (2007) / **DOI** 10.1002/mana.200510539

# Applications of operator equalities to singular integral operators and to Riemann boundary value problems

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## 1 Introduction

We denote by $[B_1, B_2]$ the set of all bounded linear operators mapping the Banach space $B_1$ into the Banach space $B_2$, $[B_1] \equiv [B_1, B_1]$.

It is known [1] that for any operator $A = X + ZY$, where $X, Y, Z \in [B_1]$ and $Z$ is an involution, $Z^2 = I$, the Gohberg–Krupnik matrix equality

$$H \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix} H^{-1} = \mathcal{D},$$

is fulfilled, where $A_1$ is an additional associated operator, $A_1 = X - ZY$, and

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ Z & -Z \end{bmatrix}, \quad H^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I & Z \\ I & -Z \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} X & ZYZ \\ Y & ZXZ \end{bmatrix}.$$

We denote the Cauchy singular integral operator along a contour $\Gamma$ by

$$(S_\Gamma \varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} \, d\tau$$

and the identity operator on $\Gamma$ by $(I_\Gamma \varphi)(t) = \varphi(t)$.

Suppose that $X = aI_\Gamma + cS_\Gamma$, $Y = (Zb)I_\Gamma + (Zd)S_\Gamma$, where $a, b, c, d$ are bounded measurable functions on $\Gamma$ and $(Z_\Gamma \varphi)(\tau) = \varphi(-\tau)$. We denote the unit circle by $\mathbb{T}$ and the real axis by $\mathbb{R}$. The matrix equality takes the form

$$H \begin{bmatrix} aI_\Gamma + bZ_\Gamma + cS_\Gamma + dZ_\Gamma S_\Gamma & 0 \\ 0 & aI_\Gamma - bZ_\Gamma + cS_\Gamma - dZ_\Gamma S_\Gamma \end{bmatrix} H^{-1} = \mathcal{D}_\Gamma, \tag{1.1}$$

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where $\mathcal{D}_\Gamma$ is a matrix characteristic singular integral operator:

$$\mathcal{D}_\mathbb{T} = \begin{bmatrix} a & b \\ (Z_\mathbb{T} b) & (Z_\mathbb{T} a) \end{bmatrix} I_\mathbb{T} + \begin{bmatrix} c & d \\ (Z_\mathbb{T} d) & (Z_\mathbb{T} c) \end{bmatrix} S_\mathbb{T},$$

$$\mathcal{D}_\mathbb{R} = \begin{bmatrix} a & b \\ (Z_\mathbb{R} b) & (Z_\mathbb{R} a) \end{bmatrix} I_\mathbb{R} + \begin{bmatrix} c & -d \\ (Z_\mathbb{R} d) & (-Z_\mathbb{R} c) \end{bmatrix} S_\mathbb{R}.$$

The operators $\mathcal{D}_\mathbb{T}$ and $\mathcal{D}_\mathbb{R}$ are different because $Z$ is an orientation-preserving shift on $\mathbb{T}$, $Z_\mathbb{T} S_\mathbb{T} = S_\mathbb{T} Z_\mathbb{T}$, but on $\mathbb{R}$ it is an orientation-reversing shift , $Z_\mathbb{R} S_\mathbb{R} = -S_\mathbb{R} Z_\mathbb{R}$.

In the article [2] we obtained a direct relation between the operator $A$ with a model involution and a matrix characteristic singular integral operator without additional associated operators: for an orientation-preserving shift it is a similarity transform $\mathcal{F}A\mathcal{F}^{-1}$ and for an orientation-reversing shift it is a transform by two invertible operators $\mathcal{H}A\mathcal{E}$. We formulate these results below.

Let $\Gamma$ and $\gamma$ be contours, and let $\gamma \subset \Gamma$. The extension of a function $f(t)$, $t \in \gamma$, to $\Gamma \setminus \gamma$ by the value zero, will be denoted by $\left(J_{\Gamma\setminus\gamma} f\right)(t)$, $t \in \Gamma$. The restriction of a function $\varphi(t)$, $t \in \Gamma$, to $\gamma$ will be denoted by $(C_\gamma \varphi)(t)$, $t \in \gamma$. The characteristic function of the set $\gamma$ given on $\Gamma$ will be denoted by $\chi_\gamma(t)$, $t \in \Gamma$.

Let $L_p(\Gamma, \rho)$ denote the space of functions on $\Gamma$ which are summable in the $p$-th power after multiplication by the weight-function $\rho$, and let $L_p^m(\Gamma, \rho)$ denote the space of $m$-dimensional vector-functions with components from $L_p(\Gamma, \rho)$.

We define $\mathcal{L} = \{z : |z| = 1,\ 0 < \arg z < 2\pi/m\}$, $(W_m \varphi)(t) = \varphi(\varepsilon_m t)$, $\varepsilon_m := \cos\frac{2\pi}{m} + i \sin\frac{2\pi}{m}$ and

$$M \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{bmatrix} = \sum_{k=1}^m W_m^{-k+1} J_{\mathbb{T}\setminus\mathcal{L}} \varphi_k, \quad M \in [L_2^m(\mathcal{L}), L_2(\mathbb{T})], \quad M^{-1}\varphi = \begin{bmatrix} C_\mathcal{L}\varphi \\ C_\mathcal{L} W_m \varphi \\ \vdots \\ C_\mathcal{L} W_m^{m-1} \varphi \end{bmatrix};$$

$$\Pi = \frac{1}{\sqrt{m}} \left[\varepsilon^{(r-1)(k-r-1)}\right]_{k,r=1}^m, \quad \Pi^{-1} = \frac{1}{\sqrt{m}} \left[\varepsilon^{(k-1)(k-r+1)}\right]_{k,r=1}^m;$$

$$V = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}, \quad \Pi^{-1} V \Pi = \Omega, \quad \Omega = \operatorname{diag}\left[1, \varepsilon^1, \dots, \varepsilon^{m-1}\right];$$

$$G(t) = \operatorname{diag}\left[1, t^1, \dots, t^{m-1}\right], \quad G^{-1}(t) = \operatorname{diag}\left[1, t^{-1}, \dots, t^{1-m}\right], \quad t \in \mathcal{L};$$

$$(N\zeta)(t) = \zeta(t^m), \quad N \in [L_2^m(\mathbb{T}), L_2^m(\mathcal{L})], \quad \left(N^{-1}\zeta\right)(t) = \zeta\left(t^{\frac{1}{m}}\right).$$

**Theorem 1.1** ([2, Theorem 2.16, p. 240]) *The singular integral operator $A$ with the shift-rotation $W_m$ at the angle $2\pi/m$ and bounded measurable coefficients,*

$$A = \sum_{k=0}^{m-1} [a_k(t) I_\mathbb{T} + b_k(t) S_\mathbb{T}] W_m^k, \quad A \in [L_2(\mathbb{T})],$$

*is similar to the matrix characteristic singular integral operator $D_\mathbb{T}$:*

$$D_\mathbb{T} = \mathcal{F}^{-1} A \mathcal{F}, \quad D_\mathbb{T} = u I_\mathbb{T} + v S_\mathbb{T}, \quad D_\mathbb{T} \in L_2^m(\mathbb{T}), \tag{1.2}$$

*where*

$$\mathcal{F} = M \Pi G N \in [L_2^m(\mathbb{T}), L_2(\mathbb{T})], \quad \mathcal{F}^{-1} = N^{-1} G^{-1} \Pi^{-1} M^{-1} \in [L_2(\mathbb{T}), L_2^m(\mathbb{T})].$$

 

*The connection between the coefficients of the operator $A$ and the coefficients of the operator $D_{\mathbb{T}}$ is given by the formulas:*

$$u(t) = \left[\frac{t^{(1-k)/m}\varepsilon^{(k-1)(k-r+1)}}{\sqrt{m}}\right]_{k,r=1}^{m} u_1\big(t^{1/m}\big) \left[\frac{\varepsilon^{(r-1)(k-r-1)}t^{(r-1)/m}}{\sqrt{m}}\right]_{k,r=1}^{m},$$
$$v(t) = \left[\frac{t^{(1-k)/m}\varepsilon^{(k-1)(k-r+1)}}{\sqrt{m}}\right]_{k,r=1}^{m} v_1\big(t^{1/m}\big) \left[\frac{\varepsilon^{(r-1)(k-r-1)}t^{(r-1)/m}}{\sqrt{m}}\right]_{k,r=1}^{m}, \quad t \in \mathbb{T},$$

*where*

$$u_1(t) = \begin{bmatrix} a_0(t) & a_1(t) & \dots & a_{m-1}(t) \\ a_{m-1}(\varepsilon t) & a_0(\varepsilon t) & \dots & a_1(\varepsilon t) \\ \vdots & \vdots & \ddots & \vdots \\ a_1(\varepsilon^{m-1}t) & a_2(\varepsilon^{m-1}t) & \dots & a_0(\varepsilon^{m-1}t) \end{bmatrix},$$

$$v_1(t) = \begin{bmatrix} b_0(t) & b_1(t) & \dots & b_{m-1}(t) \\ b_{m-1}(\varepsilon t) & b_0(\varepsilon t) & \dots & b_1(\varepsilon t) \\ \vdots & \vdots & \ddots & \vdots \\ b_1(\varepsilon^{m-1}t) & b_2(\varepsilon^{m-1}t) & \dots & b_0(\varepsilon^{m-1}t) \end{bmatrix}, \quad t \in \mathcal{L}.$$

Now we formulate a theorem for the case of an orientation reversing shift.

We denote the positive semiaxis by $\mathbb{R}_+ = (0, +\infty)$ and the negative semiaxis by $\mathbb{R}_- = (-\infty, 0)$;

$$(Q\varphi)(x) = \frac{\sqrt{\delta^2+\beta}}{x-\delta}\,\varphi[\alpha(x)], \quad \alpha(x) = \frac{\delta x + \beta}{x-\delta}, \quad \delta \in \mathbb{R}, \quad \beta \in \mathbb{R}, \quad \delta^2 + \beta > 0,$$

$Q$ is an involution, $Q^2 = I_{\mathbb{R}}$ generated by a Carleman linear-fractional orientation-reversing shift, $\alpha(\alpha(x)) \equiv x$;

$$(\Theta\varphi)(x) = \frac{x_2 - x_1}{x_2 - x}\varphi\left(\frac{x - x_1}{x_2 - x}\right), \quad (\Theta^{-1}\varphi)(x) = \frac{1}{x+1}\,\varphi\left(\frac{x_2 x + x_1}{x+1}\right),$$

where

$$x_1 = \delta - \sqrt{\delta^2+\beta}, \qquad x_2 = \delta + \sqrt{\delta^2+\beta};$$

$$(N_{\mathbb{R}_+}\varphi)(t) = \varphi\big(t^2\big), \quad \big(N_{\mathbb{R}_+}^{-1}\varphi\big)(t) = \varphi\big(\sqrt{t}\,\big);$$

$$P = \begin{bmatrix} S_{\mathbb{R}_+} + U_{1,\mathbb{R}_+} & 0 \\ 0 & I_{\mathbb{R}_+} \end{bmatrix};$$

$$\Pi^{\pm 1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$M_{\mathbb{R}_+} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{cases} \varphi_1(t), & t \in \mathbb{R}_+, \\ \varphi_2(-t), & t \in \mathbb{R}_-, \end{cases} \quad M_{\mathbb{R}_+}^{-1}\varphi = \begin{bmatrix} \varphi(t) \\ \varphi(-t) \end{bmatrix}, \quad t \in \mathbb{R}_+;$$

$$\Theta \in [L_2(\mathbb{R})], \quad P \in \big[L_2^2(\mathbb{R}_+)\big], \quad N_{\mathbb{R}_+} \in \big[L_2^2\big(\mathbb{R}_+, t^{-\frac{1}{4}}\big), L_2^2(\mathbb{R}_+)\big], \quad M_{\mathbb{R}_+} \in \big[L_2^2(\mathbb{R}_+), L_2(\mathbb{R})\big].$$

**Theorem 1.2** ([2, Theorem 3.11, p. 244]) *The singular integral operator $B$ with the involution $Q$, with bounded measurable coefficients,*

$$B_{\mathbb{R}} = aI_{\mathbb{R}} + bQ + cS_{\mathbb{R}} + dQS_{\mathbb{R}}, \quad B_{\mathbb{R}} \in [L_2(\mathbb{R})],$$

*can be reduced by invertible operators to the matrix characteristic singular integral operator $D_{\mathbb{R}_+}$:*

$$D_{\mathbb{R}_+} = \mathcal{H} B_{\mathbb{R}} \mathcal{E}, \quad D_{\mathbb{R}_+} = uI_{\mathbb{R}_+} + vS_{\mathbb{R}_+}, \quad D_{\mathbb{R}_+} \in \big[L_2^2\big(\mathbb{R}_+, t^{-\frac{1}{4}}\big)\big], \tag{1.3}$$

 

*where*

$$\mathcal{H} = N_{\mathbb{R}_+}^{-1}\Pi^{-1}M_{\mathbb{R}}^{-1}\Theta^{-1} \in \big[L_2^2(\mathbb{R}_+), L_2^2\big(\mathbb{R}_+, t^{-\frac{1}{4}}\big)\big],$$
$$\mathcal{E} = \Theta M_{\mathbb{R}}\Pi P N_{\mathbb{R}_+} \in \big[L_2^2\big(\mathbb{R}_+, t^{-\frac{1}{4}}\big), L_2^2(\mathbb{R}_+)\big].$$

*The relation between the coefficients of the operator $B_{\mathbb{R}}$ and the coefficients of the operator $D_{\mathbb{R}_+}$ is given by the formulas:*

$$u(t) = \frac{1}{2}\begin{bmatrix} (c(\zeta_+(t)) + d(\zeta_+(t))) - (c(\zeta_-(t)) + d(\zeta_-(t))) & (a(\zeta_+(t)) + b(\zeta_+(t))) - (a(\zeta_-(t)) + b(\zeta_-(t))) \\ (c(\zeta_+(t)) + d(\zeta_+(t))) + (c(\zeta_-(t)) + d(\zeta_-(t))) & (a(\zeta_+(t)) + b(\zeta_+(t))) + (a(\zeta_-(t)) + b(\zeta_-(t))) \end{bmatrix}, \tag{1.4}$$

$$v(t) = \frac{1}{2}\begin{bmatrix} (a(\zeta_+(t)) - b(\zeta_+(t))) + (a(\zeta_-(t)) - b(\zeta_-(t))) & (c(\zeta_+(t)) - d(\zeta_+(t))) + (c(\zeta_-(t)) - d(\zeta_-(t))) \\ (a(\zeta_+(t)) - b(\zeta_+(t))) - (a(\zeta_-(t)) - b(\zeta_-(t))) & (c(\zeta_+(t)) - d(\zeta_+(t))) - (c(\zeta_-(t)) - d(\zeta_-(t))) \end{bmatrix}, \tag{1.5}$$

*where*

$$\zeta_+(t) = \frac{x_2\sqrt{t} + x_1}{\sqrt{t}+1}, \quad \zeta_-(t) = \frac{-x_2\sqrt{t} + x_1}{-\sqrt{t}+1}, \quad t \in \mathbb{R}_+.$$

We will refer to formulas (1.2) and (1.3) as operator equalities. In this paper we will use the operator equalities to study the invertibility properties of singular integral operators.

In Section 2, we consider a Riemann boundary value problem with shift and piecewise constant coefficients. In Section 3, we consider a special case of the matrix characteristic singular integral operator. The coefficients of the operator are piecewise constant matrix-functions having at most four different values.

We are interested in the questions connected with solvability problems: descriptions of the kernels, conditions for the invertibility, construction of the solutions which are more detailed than the study of Fredholm properties [1], [4].

Using the operator equalities we obtain conditions for the existence and uniqueness of solution to the boundary value problem and conditions for the invertibility of the matrix characteristic operator.

## 2 Riemann boundary value problem with piecewise constant coefficients

We consider the following problem: find an analytical function $\Phi(z)$ in the strip $T = \{z : -1 \leq \mathrm{Im} z \leq +1\}$ subject to the functional relation

$$A(x)\Phi(x+i) + B(x)\Phi(x-i) + C(x)\Phi(x) = H(x), \tag{2.1}$$

where $x \in \mathbb{R}$, $\mathbb{R} = (-\infty, +\infty)$, the coefficients $A(x), B(x), C(x)$ are bounded measurable functions, and $H(x) \in L_2(\mathbb{R})$. We assume as well that $\Phi(x+i) \in L_2(\mathbb{R}), \Phi(x-i) \in L_2(\mathbb{R})$.

Our main aim in this section is to obtain conditions for the existence and uniqueness of solution to the boundary value problem for the case of piecewise constant coefficients with two different values and a point of discontinuity at $x = 0$.

We start with the formulation of a result about invertibility of characteristic singular integral operators with a certain piecewise constant matrix-function [5], [6].

Let $L_p(\mathbb{R}, \varrho) = \{f : \varrho f \in L_p\}$, $\varrho = \left(1+t^2\right)^{\nu/2} |t|^{\nu_0} |t-1|^{\nu_1}$, $1 < p < \infty$, $\nu_2 = 1 - \frac{2}{p} - \nu - \nu_0 - \nu_1$, $-\frac{1}{p} < \nu_k < 1 - \frac{1}{p}$, $k = 0, 1, 2$.

Given two non-singular constant matrices $\mathcal{A}$ and $\mathcal{B}$, following [5] we denote the arguments of the eigenvalues of $\mathcal{A}, \mathcal{A}^{-1}\mathcal{B}$ and $\mathcal{B}^{-1}$ by $2\pi\nu_{0k}(\mathcal{A},\mathcal{B})$, $2\pi\nu_{1k}(\mathcal{A},\mathcal{B})$, and $2\pi\nu_{2k}(\mathcal{A},\mathcal{B})$ $(k = 1, 2)$, respectively. In case the matrices $\mathcal{A}$ and $\mathcal{B}$ have common eigenvectors, let us agree upon attaching the same subscript $k$ to the gammas associated with the corresponding eigenvalues. If the matrices $\mathcal{A}$ and $\mathcal{B}$ share (up to linear dependence) exactly one common eigenvector, we shall label the corresponding gamma by the subscript $k = 2$. We introduce the numbers

$$l_k(\mathcal{A},\mathcal{B}) = \sum_{j=0}^{2}\left(\nu_{jk}(\mathcal{A},\mathcal{B}) + [\delta_{jk}(\mathcal{A},\mathcal{B})]\right), \quad \delta_{jk}(\mathcal{A},\mathcal{B}) = \frac{1}{p} + \nu_j - \nu_{jk}(\mathcal{A},\mathcal{B}), \qquad k = 1, 2, \quad j = 0, 1, 2. \tag{2.2}$$

 

By [x] we mean the integral part of x.

**Theorem 2.1** ([5, Corollary 2, p. 248]) *For the operator* $R(G_{\mathbb{R}}) = P^+_{\mathbb{R}} + G_{\mathbb{R}} P^-_{\mathbb{R}}$*, generated by a matrix-function* $G_{\mathbb{R}} = E_2\chi_{(-\infty,0)} + \mathcal{A}\chi_{(0,1)} + \mathcal{B}\chi_{(1,+\infty)}$*, to be invertible in* $L^2_p(\mathbb{R}, \varrho)$*, it is necessary and sufficient that the constant matrices* $\mathcal{A}, \mathcal{B}$ *are non-singular, that the numbers* $\delta_{jk}(\mathcal{A}, \mathcal{B})$ *are non-integer, and that at least one of the following conditions hold:*

(i) $\mathcal{A}$ *and* $\mathcal{B}$ *have no common eigenvectors and* $l_1(\mathcal{A}, \mathcal{B}) = -l_2(\mathcal{A}, \mathcal{B})$*;*

(ii) $\mathcal{A}$ *and* $\mathcal{B}$ *do not commute, possess a common eigenvector, and* $l_1(\mathcal{A}, \mathcal{B}) = -l_2(\mathcal{A}, \mathcal{B}) \geq 0$*;*

(iii) $\mathcal{A}$ *and* $\mathcal{B}$ *commute and* $l_1(\mathcal{A}, \mathcal{B}) = l_2(\mathcal{A}, \mathcal{B}) = 0$.

Let $A_\pm, B_\pm, C_\pm$ be constants.

**Theorem 2.2** *Let* $3A_+ + B_+ \neq 0$ *and* $3A_- + B_- \neq 0$*. If the matrices*

$$\mathcal{A} = -\frac{2}{3A_+ + B_+} \begin{bmatrix} 3A_+ + A_- + B_+ + 3B_- + 4i(C_+ + C_-) & 3A_+ - A_- + B_+ - 3B_- + 4i(C_+ + C_-) \\ -3A_+ + A_- - B_+ + 3B_- + 4i(C_+ + C_-) & -3A_+ - A_- - B_+ - 3B_- + 4i(C_+ + C_-) \end{bmatrix} \tag{2.3}$$

*and*

$$\mathcal{B} = -\frac{2}{3A_- + B_-} \begin{bmatrix} A_+ + 3A_- + 3B_+ + B_- + 4i(C_+ + C_-) & -A_+ + 3A_- - 3B_+ + B_- + 4i(C_+ + C_-) \\ A_+ - 3A_- + 3B_+ - B_- + 4i(C_+ + C_-) & -A_+ - 3A_- - 3B_+ - B_- + 4i(C_+ + C_-) \end{bmatrix} \tag{2.4}$$

*satisfy the conditions*

(a) $\det \mathcal{A} \neq 0$, $\det \mathcal{B} \neq 0$,

(b) *for* $k = 1, 2$, *and* $j = 0, 1, 2$, *the numbers* $\delta_{jk}$ *are not integers,*

(c) *one of the three conditions* (i), (ii), (iii) *is fulfilled,*

*then the Riemann boundary value problem with shift* (2.1)*:*

$$A(x)\Phi(x + i) + B(x)\Phi(x - i) + C(x)\Phi(x) = H(x), \quad x \in \mathbb{R},$$

*and piecewise constant coefficients*

$$\begin{aligned} A(x) &= A_-\chi_{\mathbb{R}_-}(x) + A_+\chi_{\mathbb{R}_+}(x), \\ B(x) &= B_-\chi_{\mathbb{R}_-}(x) + B_+\chi_{\mathbb{R}_+}(x), \\ C(x) &= C_-\chi_{\mathbb{R}_-}(x) + C_+\chi_{\mathbb{R}_+}(x), \end{aligned}$$

*admits a unique solution.*

P r o o f. To prove the theorem we follow the schema of [3].

According to [4] the boundary value problem (2.1) can be transformed to the following integral equation with endpoint singularities considered on the space $L_2(\mathcal{T})$,

$$a_{\mathcal{T}}(\xi) w_{\mathcal{T}}(\xi) + \frac{c_{\mathcal{T}}(\xi)}{\pi i} \int_{\mathcal{T}} \frac{w_{\mathcal{T}}(\tau)\, d\tau}{\tau - \xi} - \frac{d_{\mathcal{T}}(\xi)}{\pi i} \int_{\mathcal{T}} \frac{w_{\mathcal{T}}(\tau)\, d\tau}{1 - \xi\tau} = g_{\mathcal{T}}(\xi), \quad \mathcal{T} = (-1, 1), \tag{2.5}$$

where

$$\begin{aligned} a_{\mathcal{T}}(\xi) &= \pi\, \frac{A([\gamma(\xi)]) + B([\gamma(\xi)])}{2}, \\ c_{\mathcal{T}}(\xi) &= \pi\, \frac{A([\gamma(\xi)]) - B([\gamma(\xi)])}{4}, \\ d_{\mathcal{T}}(\xi) &= -\pi i C[\gamma(\xi)], \\ g_{\mathcal{T}}(\xi) &= \frac{\pi H[\gamma(\xi)]}{\sqrt{1 - \xi^2}}, \\ \gamma(\xi) &= \frac{1}{\pi} \ln \frac{1 + \xi}{1 - \xi}. \end{aligned}$$

The solutions of problem (2.1) and Equation (2.5) are connected by

$$w_{\mathcal{T}}(\xi) = \frac{Y[\gamma(\xi)]}{\sqrt{1-\xi^2}}, \quad Y(x) = (F\omega)(x), \quad \omega(x) = [\exp(x) + \exp(-x)]\big(F^{-1}\Phi\big)(x),$$

where $F$ and $F^{-1}$ are the direct and inverse Fourier transformation.

Taking into account that

$$\frac{1}{\pi i}\int_{\mathcal{T}} \frac{w_{\mathcal{T}}(\tau)}{1-\xi\tau}\, d\tau = -(C_{\mathcal{T}} Q_{\mathcal{T}} S_{\mathbb{R}} J_{\mathbb{R}\setminus\mathcal{T}} w_{\mathcal{T}})(\xi), \quad \xi \in \mathcal{T},$$

where $Q$ is an involution generated by the points $x_1 = -1$, $x_2 = +1$,

$$(Q_{\mathcal{T}}\varphi)(x) = \frac{1}{x}\,\varphi[\alpha(x)], \quad \alpha(x) = \frac{1}{x}, \quad x \in \mathbb{R},$$

rewrite (2.5) in the form

$$(K_{\mathcal{T}} w_{\mathcal{T}})(\xi) = g_{\mathcal{T}}(\xi), \quad K_{\mathcal{T}} = a_{\mathcal{T}} I_{\mathcal{T}} + c_{\mathcal{T}} S_{\mathcal{T}} + d_{\mathcal{T}} C_{\mathcal{T}} Q_{\mathcal{T}} S_{\mathbb{R}} J_{\mathbb{R}\setminus\mathcal{T}}, \quad K_{\mathcal{T}} \in [L_2(\mathcal{T})].$$

Here

$$a_{\mathcal{T}}(\xi) = \frac{1}{2}\,\pi[(A_- + B_-)\chi_{(-1,0)}(\xi) + (A_+ + B_+)\chi_{(0,1)}(\xi)],$$

$$c_{\mathcal{T}}(\xi) = \frac{1}{4}\,\pi[(A_- - B_-)\chi_{(-1,0)}(\xi) + (A_+ - B_+)\chi_{(0,1)}(\xi)],$$

$$d_{\mathcal{T}}(\xi) = -\pi i[C_-\chi_{(-1,0)}(\xi) + C_+\chi_{(0,1)}(\xi)], \quad \xi \in \mathcal{T}.$$

Extend the operator $K_{\mathcal{T}}$ to the whole real axis $\mathbb{R}$:

$$K^1_{\mathbb{R}} = \widetilde{a}_{\mathbb{R}} I_{\mathbb{R}} + \widetilde{c}_{\mathbb{R}} S_{\mathbb{R}} + \widetilde{d}_{\mathbb{R}} T_{\mathbb{R}} S_{\mathbb{R}},$$

where

$$\widetilde{a}_{\mathbb{R}} = (\chi_{\mathbb{R}\setminus\mathcal{T}} + J_{\mathbb{R}\setminus\mathcal{T}} a_{\mathcal{T}}), \quad \widetilde{c}_{\mathbb{R}} = (J_{\mathbb{R}\setminus\mathcal{T}} c_{\mathcal{T}}), \quad \widetilde{d}_{\mathbb{R}} = (J_{\mathbb{R}\setminus\mathcal{T}} d_{\mathcal{T}}).$$

The operator $K_{\mathcal{T}}$ is invertible in the space $L_2(\mathcal{T})$ if and only if the operator $K^1_{\mathbb{R}}$ is invertible in the space $L_2(\mathbb{R})$.

Note [3] that if $K_{\mathcal{T}}^{-1}$ is the inverse operator of $K_{\mathcal{T}}$, then the operator

$$\big(K^1_{\mathbb{R}}\big)^{-1} = \chi_{\mathbb{R}\setminus\mathcal{T}} I_{\mathbb{R}} + J_{\mathbb{R}\setminus\mathcal{T}} K_{\mathcal{T}}^{-1} C_{\mathcal{T}}\big(I_{\mathbb{R}} - K^1_{\mathbb{R}}\chi_{\mathbb{R}\setminus\mathcal{T}} I_{\mathbb{R}}\big)$$

is the inverse operator of the operator $K^1_{\mathbb{R}}$, and if $\big(K^1_{\mathbb{R}}\big)^{-1}$ is the inverse operator of $K^1_{\mathbb{R}}$, then the operator

$$K_{\mathcal{T}}^{-1} = C_{\mathcal{T}}\big(K^1_{\mathbb{R}}\big)^{-1} J_{\mathbb{R}\setminus\mathcal{T}}$$

is the inverse operator of the operator $K_{\mathcal{T}}$.

Applying the operator equality (1.3) to the equation $\big(K^1_{\mathbb{R}}\varphi\big)(x) = J_{\mathbb{R}\setminus\mathcal{T}} g_{\mathcal{T}}$, we obtain the equivalent matrix characteristic singular integral equation

$$\widetilde{D}_{\mathbb{R}_+}\psi_{\mathbb{R}_+} = \widetilde{g}_{\mathbb{R}_+}, \quad \widetilde{D}_{\mathbb{R}_+} = \mathcal{H} K^1_{\mathbb{R}} \mathcal{E} = \widetilde{u}_{\mathbb{R}_+} I_{\mathbb{R}_+} + \widetilde{v}_{\mathbb{R}_+} S_{\mathbb{R}_+}, \quad \widetilde{D}_{\mathbb{R}_+} \in \big[L^2_2\big(\mathbb{R}_+, t^{-\frac{1}{4}}\big)\big], \tag{2.6}$$

where

$$\widetilde{g}_{\mathbb{R}_+} = \mathcal{H} J_{\mathbb{R}\setminus\mathcal{T}} g_{\mathcal{T}}, \quad \widetilde{g}_{\mathbb{R}_+} \in L^2_2\big(\mathbb{R}_+, t^{-\frac{1}{4}}\big),$$

and the coefficients have the form

$$\widetilde{u}_{\mathbb{R}_+}(t) = \frac{1}{2}\begin{bmatrix} \widetilde{u}_{11}(t) & \widetilde{u}_{12}(t) \\ \widetilde{u}_{21}(t) & \widetilde{u}_{22}(t) \end{bmatrix}, \quad \widetilde{v}_{\mathbb{R}_+}(t) = \frac{1}{2}\begin{bmatrix} \widetilde{v}_{11}(t) & \widetilde{v}_{12}(t) \\ \widetilde{v}_{21}(t) & \widetilde{v}_{22}(t) \end{bmatrix}, \quad t \in \mathbb{R}_+,$$

with

$$\widetilde{u}_{11}(t) = \widetilde{u}_{21}(t) = \left[\frac{1}{4}\,\pi(A_- - B_-) - \pi i C_-\right]\chi_{(1,\infty)}(t) + \left[\frac{1}{4}\,\pi(A_+ - B_+) - \pi i C_+\right]\chi_{(0,1)}(t),$$
$$\widetilde{u}_{12}(t) = \left[\frac{1}{2}\pi(A_- + B_-) - 1\right]\chi_{(1,\infty)}(t) + \left[\frac{1}{2}\pi(A_+ + B_+) - 1\right]\chi_{(0,1)}(t),$$
$$\widetilde{u}_{22}(t) = \left[\frac{1}{2}\pi(A_- + B_-) + 1\right]\chi_{(1,\infty)}(t) + \left[\frac{1}{2}\,\pi(A_+ + B_+) + 1\right]\chi_{(0,1)}(t),$$
$$\widetilde{v}_{11}(t) = \widetilde{u}_{22}(t),$$
$$\widetilde{v}_{21}(t) = \widetilde{u}_{12}(t),$$
$$\widetilde{v}_{12}(t) = \widetilde{v}_{22}(t) = \left[\frac{1}{4}\,\pi(A_- - B_-) + \pi i C_-\right]\chi_{(1,\infty)}(t) + \left[\frac{1}{4}\,\pi(A_+ - B_+) + \pi i C_+\right]\chi_{(0,1)}(t).$$

The solutions of Equations (2.5) and (2.6) are connected by

$$\psi_{\mathbb{R}_+}(t) = \big(\mathcal{E}^{-1}(J_{\mathbb{R}\setminus\mathcal{T}}\omega_{\mathcal{T}})\big)(t).$$

Note that the coefficients $\widetilde{u}_{\mathbb{R}_+}$ and $\widetilde{u}_{\mathbb{R}_+}$ are piecewise constant matrix-functions with two values and a point of discontinuity at $x = 1$ on the contour $\mathbb{R}_+$.

The operator $\widetilde{D}_{\mathbb{R}_+}$ is extended on $\mathbb{R}_- = (-\infty, 0)$

$$\widetilde{D}^1_{\mathbb{R}}\psi = J_{\mathbb{R}_-}\widetilde{g}_{\mathbb{R}_+}, \quad \widetilde{D}^1_{\mathbb{R}} = (\chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}\widetilde{u}_{\mathbb{R}_+})I_{\mathbb{R}} + (J_{\mathbb{R}_-}\widetilde{v}_{\mathbb{R}_+})S_{\mathbb{R}}, \quad \widetilde{D}^1_{\mathbb{R}} \in \big[L^2_2\big(\mathbb{R}, t^{-\frac{1}{4}}\big)\big].$$

The operator $\widetilde{D}_{\mathbb{R}_+}$ is invertible on the space $L^2_2\big(\mathbb{R}_+, t^{-\frac{1}{4}}\big)$ if and only if the operator $\widetilde{D}^1_{\mathbb{R}}$ is invertible on the space $L^2_2\big(\mathbb{R}, t^{-\frac{1}{4}}\big)$.

Rewrite the operator $\widetilde{D}^1_{\mathbb{R}}$ using the projections $P^+_{\mathbb{R}} = \frac{1}{2}\,(I_{\mathbb{R}} + S_{\mathbb{R}})$ and $P^-_{\mathbb{R}} = \frac{1}{2}\,(I_{\mathbb{R}} - S_{\mathbb{R}})$:

$$\widetilde{D}^1_{\mathbb{R}} = R(\mathcal{U}, \mathcal{V}) = \mathcal{U}P^+_{\mathbb{R}} + \mathcal{V}P^-_{\mathbb{R}},$$

where

$$\mathcal{U} = E_2\chi_{\mathbb{R}_-} + \mathcal{U}_{(0,1)}\chi_{(0,1)} + \mathcal{U}_{(1,\infty)}\chi_{(1,+\infty)},$$
$$\mathcal{V} = E_2\chi_{\mathbb{R}_-} + \mathcal{V}_{(0,1)}\chi_{(0,1)} + \mathcal{V}_{(1,\infty)}\chi_{(1,+\infty)},$$
$$E_2 = \operatorname{diag}[1,1],$$

$$\mathcal{U}_{(0,1)} = \frac{1}{2}\begin{bmatrix} \frac{\pi}{2}(A_+ + B_+) + \frac{\pi}{4}(A_+ - B_+) - \pi C_+ + 1 & \frac{\pi}{2}(A_+ + B_+) + \frac{\pi}{4}(A_+ - B_+) + \pi C_+ - 1 \\ \frac{\pi}{2}(A_+ + B_+) + \frac{\pi}{4}(A_+ - B_+) - \pi C_+ - 1 & \frac{\pi}{2}(A_+ + B_+) + \frac{\pi}{4}(A_+ - B_+) + \pi C_+ + 1 \end{bmatrix},$$

$$\mathcal{U}_{(1,+\infty)} = \frac{1}{2}\begin{bmatrix} \frac{\pi}{2}(A_- + B_-) + \frac{\pi}{4}(A - B_-) - \pi C_- + 1 & \frac{\pi}{2}(A_- + B_-) + \frac{\pi}{4}(A - B_-) + \pi C_- - 1 \\ \frac{\pi}{2}(A_- + B_-) + \frac{\pi}{4}(A - B_-) - \pi C_- - 1 & \frac{\pi}{2}(A_- + B_-) + \frac{\pi}{4}(A - B_-) + \pi C_- + 1 \end{bmatrix},$$

$$\mathcal{V}_{(0,1)} = \frac{1}{2}\begin{bmatrix} \frac{\pi}{2}(A_- + B_-) - \frac{\pi}{4}(A - B_-) + \pi C_- + 1 & -\frac{\pi}{2}(A_- + B_-) + \frac{\pi}{4}(A - B_-) + \pi C_- + 1 \\ \frac{\pi}{2}(A_- + B_-) - \frac{\pi}{4}(A - B_-) + \pi C_- - 1 & -\frac{\pi}{2}(A_- + B_-) + \frac{\pi}{4}(A - B_-) + \pi C_- - 1 \end{bmatrix},$$

$$\mathcal{V}_{(1,+\infty)} = \frac{1}{2}\begin{bmatrix} \frac{\pi}{2}(A_+ + B_+) - \frac{\pi}{4}(A_+ - B_+) + \pi C_+ + 1 & -\frac{\pi}{2}(A_+ + B_+) + \frac{\pi}{4}(A_+ - B_+) + \pi C_+ + 1 \\ \frac{\pi}{2}(A_+ + B_+) - (A_+ - B_+) + \pi C_+ - 1 & \frac{\pi}{2}(A_+ + B_+) + \frac{\pi}{4}(A_+ - B_+) + \pi C_+ - 1 \end{bmatrix}.$$

We assume that $\det(\widetilde{u}_{\mathbb{R}} + \widetilde{v}_{\mathbb{R}}) \neq 0$, or $\det(E_2\chi_{\mathbb{R}_-} + \mathcal{U}_{(0,1)}\chi_{(0,1)} + \mathcal{U}_{(1,\infty)}\chi_{(1,+\infty)}) \neq 0$, or

$$\det\mathcal{U}_{(0,1)} = \frac{2}{3A_+ + B_+} \neq 0, \quad \det\mathcal{U}_{(1,\infty)} = \frac{2}{3A_- + B_-} \neq 0.$$

Having calculated the matrices $\mathcal{A} = \mathcal{U}^{-1}_{(0,1)}\mathcal{V}_{(0,1)}$ and $\mathcal{B} = \mathcal{U}^{-1}_{(1,+\infty)}\mathcal{V}_{(1,+\infty)}$, we obtain

$$\begin{aligned}\mathcal{G}_{\mathbb{R}} &= (\chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}(\widetilde{u}_{\mathbb{R}_+} + \widetilde{v}_{\mathbb{R}_+}))^{-1}(\chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}(\widetilde{u}_{\mathbb{R}_+} - \widetilde{v}_{\mathbb{R}_+}))\\ &= \mathcal{U}^{-1}\mathcal{V}\\ &= \chi_{\mathbb{R}_-}E_2 + \chi_{(0,1)}\mathcal{A} + \chi_{(1,+\infty)}\mathcal{B}).\end{aligned}$$

Here the matrices $\mathcal{A}$ and $\mathcal{B}$ are given by (2.3) and (2.4) respectively.

Consider the operator $R(\mathcal{G}_{\mathbb{R}}) = P^+_{\mathbb{R}} + \mathcal{G}P^-_{\mathbb{R}}$ acting on the space $L^2_p(\mathbb{R}, \varrho)$, $\varrho(x) = |x|^{-1/4}$ .

The matrix $\mathcal{G}_{\mathbb{R}}(x)$, $x \in \mathbb{R}$, is a piecewise constant matrix-function with three values and points of discontinuity at $x = 0$, $x = 1$. Applying [5, Corollary 2] to the operator $\mathcal{R}(\mathcal{G}_{\mathbb{R}})$, we complete the proof of the theorem. □

## 3 Invertibility of matrix characteristic singular integral operators with coefficients of a special structure

Let us consider the weight space $L_p(\mathbb{R}, \rho_W)$, $p \geq 1$, $(\rho_W)(x) = \prod_{j=1}^4 |x - x_j|^{\mu_j}$, $x_1 = -1$, $x_2 = 1$, $x_3 = 0$, $x_4 = i$ with the norm $\|f\|_{L_p(\mathbb{R},\rho_W)} = \|\rho_W f\|_{L_p(\mathbb{R})}$, assuming that the following conditions hold:

$$\frac{-1}{p} < \mu_j < \frac{p-1}{p}, \quad j = 1,2,3; \qquad \frac{-1}{p} < \sum_{j=1}^4 \mu_j < \frac{p-1}{p}; \qquad \mu_3 = -\sum_{j=1}^4 \mu_j + \frac{p-2}{p}. \tag{3.1}$$

In the space $L_p(\mathbb{R}, \rho_W)$ consider the operator $\mathcal{D}_{\mathbb{R}} = uI_{\mathbb{R}} + vS_{\mathbb{R}}$ with coefficients which are piecewise constant matrix-functions with three points of discontinuity at $x = -1$, $x = 0$, $x = 1$:

$$u = \begin{bmatrix} a_{-2} & b_{-2}\\ b_{+2} & a_{+2}\end{bmatrix}\chi_{(-\infty,-1)} + \begin{bmatrix} a_{-1} & b_{-1}\\ b_{+1} & a_{+1}\end{bmatrix}\chi_{-1,0)} + V\begin{bmatrix} a_{-1} & b_{-1}\\ b_{+1} & a_{+1}\end{bmatrix}V\chi_{(0,+1)} + V\begin{bmatrix} a_{-2} & b_{-2}\\ b_{+2} & a_{+2}\end{bmatrix}V\chi_{(1,\infty)},$$

$$v = \begin{bmatrix} c_{-2} & -d_{-2}\\ d_{+2} & -c_{+2}\end{bmatrix}\chi_{(-\infty,-1)} + \begin{bmatrix} c_{-1} & -d_{-1}\\ d_{+1} & -c_{+1}\end{bmatrix}\chi_{(-1,0)} - V\begin{bmatrix} c_{-1} & -d_{-1}\\ d_{+1} & -c_{+2}\end{bmatrix}V\chi_{(0,1)} - V\begin{bmatrix} c_{-2} & -d_{-2}\\ d_{+2} & -c_{+2}\end{bmatrix}V\chi_{(1,\infty)},$$

where

$$V = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}.$$

From (3.1) it follows that $S_{\mathbb{R}} \in [L_p(\mathbb{R}, \rho_W)]$ and $W_{\mathbb{R}} \in [L_p(\mathbb{R}, \rho_W)]$, $(W_{\mathbb{R}}\varphi)(x) = \varphi(-x)$.

In this section conditions for the invertibility of the operator $\mathcal{D}_{\mathbb{R}}$ in the space $L_p(\mathbb{R}, \rho_W)$ are obtained.

We introduce the functions

$$\begin{aligned}a(x) &= a_{-2}\chi_{(-\infty,-1)}(x) + a_{-1}\chi_{(-1,0)}(x) + a_{+1}\chi_{(0,1)}(x) + a_{+2}\chi_{(1,+\infty)}(x),\\ b(x) &= b_{-2}\chi_{(-\infty,-1)}(x) + b_{-1}\chi_{(-1,0)}(x) + b_{+1}\chi_{(0,1)}(x) + b_{+2}\chi_{(1,+\infty)}(x),\\ c(x) &= c_{-2}\chi_{(-\infty,-1)}(x) + c_{-1}\chi_{(-1,0)}(x) + c_{+1}\chi_{(0,1)}(x) + c_{+2}\chi_{(1,+\infty)}(x),\\ d(x) &= d_{-2}\chi_{(-\infty,-1)}(x) + d_{-1}\chi_{(-1,0)}(x) + d_{+1}\chi_{(0,1)}(x) + d_{+2}\chi_{(1,+\infty)}(x),\end{aligned}$$

and construct the matrices

$$\begin{aligned}\mathcal{A}^{\pm} &= -\det{}^{-1}\begin{bmatrix} a_{-1}+c_{-1} & \mp b_{-1}\pm d_{-1}\\ \mp b_{-2}\mp d_{-2} & a_{-2}-c_{-2}\end{bmatrix}\Pi^{-1}\begin{bmatrix} a_{-2}-c_{-2} & \pm b_{-1}\mp d_{-1}\\ \pm b_{-2}\pm d_{-2} & a_{-1}+c_{-1}\end{bmatrix}\begin{bmatrix} a_{-1}-c_{-1} & \mp b_{-1}\mp d_{-1}\\ \mp b_{-2}\pm d_{-2} & a_{-2}+c_{-2}\end{bmatrix}\Pi\Omega,\\ \mathcal{B}^{\pm} &= -\det{}^{-1}\begin{bmatrix} a_{+1}+c_{+1} & \mp b_{+1}\pm d_{+1}\\ \mp b_{+2}\mp d_{+2} & a_{+2}-c_{+2}\end{bmatrix}\Pi^{-1}\begin{bmatrix} a_{+2}-c_{+2} & \pm b_{+1}\mp d_{+1}\\ \pm b_{+2}\pm d_{+2} & a_{+1}+c_{+1}\end{bmatrix}\begin{bmatrix} a_{+1}-c_{+1} & \mp b_{+1}\mp d_{+1}\\ \mp b_{+2}\pm d_{+2} & a_{+2}+c_{+2}\end{bmatrix}\Pi\Omega,\end{aligned} \tag{3.2}$$

where

$$\Pi = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1\\ 1 & -1\end{bmatrix}, \quad \Omega = \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}.$$

Using definitions (2.2) of Section 2, we introduce the constants $l^{\pm}_k = l_k(\mathcal{A}^{\pm}, \mathcal{B}^{\pm})$, $\delta^{\pm}_{jk} = \delta_{jk}(\mathcal{A}^{\pm}, \mathcal{B}^{\pm})$.

 

**Theorem 3.1** *Let*

$$\det \begin{bmatrix} a_{-1}+c_{-1} & \mp b_{-1} \pm d_{-1} \\ \mp b_{-2} \mp d_{-2} & a_{-2}-c_{-2} \end{bmatrix} \neq 0$$

*and let*

$$\det \begin{bmatrix} a_{+1}+c_{+1} & \mp b_{+1} \pm d_{+1} \\ \mp b_{+2} \mp d_{+2} & a_{+2}-c_{+2} \end{bmatrix} \neq 0.$$

*In order that the operator $D_{\mathbb{R}}$,*

$$\mathcal{D}_{\mathbb{R}} = \begin{bmatrix} a(x) & b(x) \\ b(-x) & a(-x) \end{bmatrix} I_{\mathbb{R}} + \begin{bmatrix} c(x) & -d(x) \\ d(-x) & -c(-x) \end{bmatrix} S_{\mathbb{R}},$$

*with piecewise constant coefficients and points of discontinuity at $x=-1$, $x=0$, $x=1$, is invertible on the space $L_p(\mathbb{R},\rho_W)$, it is necessary and sufficient that the matrices $\mathcal{A}^+$, $\mathcal{B}^+$ and $\mathcal{A}^-$, $\mathcal{B}^-$ have the following properties:*

(a) $\det \mathcal{A}^+ \neq 0$, $\det \mathcal{B}^+ \neq 0$ *and* $\det \mathcal{A}^- \neq 0$, $\det \mathcal{B}^- \neq 0$*;*

(b) *for $k=1,2$, and $j=0,1,2$, the numbers $\delta^+_{jk}$ and $\delta^-_{jk}$ are not integers;*

(c) *for the pair $\mathcal{A}^+$, $\mathcal{B}^+$ and for the pair $\mathcal{A}^-$, $\mathcal{B}^-$ one of the following three conditions* (i), (ii), (iii) *is fulfilled.*

P r o o f. By the Gohberg–Krupnik matrix equality (1.1)

$$\frac{1}{2}\begin{bmatrix} I_{\mathbb{R}} & I_{\mathbb{R}} \\ W_{\mathbb{R}} & -W_{\mathbb{R}} \end{bmatrix}\begin{bmatrix} aI_{\mathbb{R}}+bW_{\mathbb{R}}I_{\mathbb{R}}+cS_{\mathbb{R}}+dW_{\mathbb{R}}S_{\mathbb{R}} & 0 \\ 0 & aI_{\mathbb{R}}-bW_{\mathbb{R}}I_{\mathbb{R}}+cS_{\mathbb{R}}-dW_{\mathbb{R}}S_{\mathbb{R}} \end{bmatrix}\begin{bmatrix} I_{\mathbb{R}} & W_{\mathbb{R}} \\ I_{\mathbb{R}} & -W_{\mathbb{R}} \end{bmatrix} = \mathcal{D}_{\mathbb{R}},$$

that the singular integral operator $\mathcal{D}_{\mathbb{R}}$ is invertible on the space $L_p(\mathbb{R},\rho_W)$, if and only if the operators $B = B_+ = aI_{\mathbb{R}} + bI_{\mathbb{R}} + cS_{\mathbb{R}} + dQS_{\mathbb{R}}$ and $B_- = aI_{\mathbb{R}} - bI_{\mathbb{R}} + cS_{\mathbb{R}} - dQS_{\mathbb{R}}$ are invertible operators on the space $L_p(\mathbb{R},\rho_W)$.

Applying the operator equality (1.3) to $B^+$ and $B^-$, we have

$$\mathcal{D}^{\pm}_{\mathbb{R}_+} = \mathcal{H}B^{\pm}\mathcal{F} = u^{\pm}_{\mathbb{R}_+} I_{\mathbb{R}_+} + v^{\pm}_{\mathbb{R}_+} S_{\mathbb{R}_+}, \quad \mathcal{D}^{\pm}_{\mathbb{R}_+} \in \left[L^2_p(\mathbb{R}_+,\varrho)\right].$$

The weight $\rho_W$ is transformed to

$$\varrho(x) = |x|^{\nu_0}|x-1|^{\nu_1}|x-i|^{\nu}, \quad \nu_0 = \frac{1}{2}\left(\mu_1 - \frac{1}{p}\right), \quad \nu_1 = \mu_3, \quad \nu = \frac{1}{2}\,\mu_4.$$

From formulas (1.4), (1.5) the coefficients of the operator $\mathcal{D}^{\pm}_{\mathbb{R}_+}$ are

$$\begin{aligned} u^{\pm}_{\mathbb{R}_+}(t) &= \frac{1}{2}\begin{bmatrix} (c_{-1}\pm d_{-1})-(c_{-2}\pm d_{-2}) & (a_{-1}\pm b_{-1})-(a_{-2}\pm b_{-2}) \\ (c_{-1}\pm d_{-1})+(c_{-2}\pm d_{-2}) & (a_{-1}\pm b_{-1})+(a_{-2}\pm b_{-2}) \end{bmatrix}\chi_{(0,1)}(t) \\ &\quad + \frac{1}{2}\begin{bmatrix} (c_{+1}\pm d_{+1})-(c_{+2}\pm d_{+2}) & (a_{+1}\pm b_{+1})-(a_{+2}\pm b_{+2}) \\ (c_{+1}\pm d_{+1})+(c_{+2}\pm d_{+2}) & (a_{+1}\pm b_{+1})+(a_{+2}\pm b_{+2}) \end{bmatrix}\chi_{(1,\infty)}(t), \\ v^{\pm}_{\mathbb{R}_+}(t) &= \frac{1}{2}\begin{bmatrix} (a_{-1}\mp b_{-1})+(a_{-2}\mp b_{-2}) & (c_{-1}\mp d_{-1})+(c_{-2}\mp d_{-2}) \\ (a_{-1}\mp b_{-1})-(a_{-2}\mp b_{-2}) & (c_{-1}\mp d_{-1})-(c_{-2}\mp d_{-2}) \end{bmatrix}\chi_{(0,1)}(t) \\ &\quad + \frac{1}{2}\begin{bmatrix} (a_{+1}\mp b_{+1})+(a_{+2}\mp b_{+2}) & (c_{+1}\mp d_{+1})+(c_{+2}\mp d_{+2}) \\ (a_{+1}\mp b_{+1})-(a_{+2}\mp b_{+2}) & (c_{+1}\mp d_{+1})-(c_{+2}\mp d_{+2}) \end{bmatrix}\chi_{(1,\infty)}(t). \end{aligned}$$

Extend the operator $\mathcal{D}^{\pm}_{\mathbb{R}_+}$ to the entire real axes $\mathcal{D}^{\pm}_{\mathbb{R}} = J_{\mathbb{R}_+}C_{\mathbb{R}_-} + J_{\mathbb{R}_-}\mathcal{D}^{\pm}_{\mathbb{R}_+}C_{\mathbb{R}_+}$, $\mathcal{D}^{\pm}_{\mathbb{R}} \in [L^2_p(\mathbb{R},\varrho)]$, and rewrite $\mathcal{D}^{\pm}_{\mathbb{R}}$ using the projections

$$\mathcal{D}^{\pm}_{\mathbb{R}} = \mathcal{U}^{\pm}_{\mathbb{R}}P^{+}_{\mathbb{R}} + \mathcal{V}^{\pm}_{\mathbb{R}}P^{-}_{\mathbb{R}},$$

where

$$\mathcal{U}_{\mathbb{R}}^{\pm} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}\left(u_{\mathbb{R}}^{\pm} + v_{\mathbb{R}_+}^{\pm}\right), \quad \mathcal{V}_{\mathbb{R}}^{\pm} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}\left(u_{\mathbb{R}}^{\pm} - v_{\mathbb{R}_+}^{\pm}\right).$$

The matrices $\mathcal{U}_{\mathbb{R}}^{\pm} = u_{\mathbb{R}_+}^{\pm}(t) + v_{\mathbb{R}_+}^{\pm}(t)$ and $\mathcal{V}_{\mathbb{R}}^{\pm} = u_{\mathbb{R}_+}^{\pm}(t) - v_{\mathbb{R}_+}^{\pm}(t)$ have the following form

$$\begin{aligned}\mathcal{U}_{\mathbb{R}}^{\pm} &= \chi_{\mathbb{R}_-} \\ &\quad + J_{\mathbb{R}_-}\Pi\left\{\begin{bmatrix} a_{-1}+c_{-1} & \mp b_{-1}\pm d_{-1} \\ \mp b_{-2}\mp d_{-2} & a_{-2}-c_{-2}\end{bmatrix}\chi_{(0,1)} + \begin{bmatrix} a_{+1}+c_{+1} & \mp b_{+1}\pm d_{+1} \\ \mp b_{+2}\mp d_{+2} & a_{+2}-c_{+2}\end{bmatrix}\chi_{(1,\infty)}\right\}\Pi,\end{aligned}$$

$$\begin{aligned}\mathcal{V}_{\mathbb{R}}^{\pm} &= \chi_{\mathbb{R}_-} \\ &\quad - J_{\mathbb{R}_-}\Pi\left\{\begin{bmatrix} a_{-1}-c_{-1} & \mp b_{-1}\mp d_{-1} \\ \mp b_{-2}\pm d_{-2} & a_{-2}+c_{-2}\end{bmatrix}\chi_{(0,1)} + \begin{bmatrix} a_{+1}-c_{+1} & \mp b_{+1}\mp d_{+1} \\ \mp b_{+2}\pm d_{+2} & a_{+2}+c_{+2}\end{bmatrix}\chi_{(1,\infty)}\right\}\Pi\Omega.\end{aligned}$$

We assume that $\det\left[u_{\mathbb{R}_+}^{\pm}(t) + v_{\mathbb{R}_+}^{\pm}(t)\right] \neq 0$, or, rewriting in an equivalent form,

$$\det\left\{\begin{bmatrix} a_{-1}+c_{-1} & \mp b_{-1}\pm d_{-1} \\ \mp b_{-2}\mp d_{-2} & a_{-2}-c_{-2}\end{bmatrix}\chi_{(0,1)} + \begin{bmatrix} a_{+1}+c_{+1} & \mp b_{+1}\pm d_{+1} \\ \mp b_{+2}\mp d_{+2} & a_{+2}-c_{+2}\end{bmatrix}\chi_{(1,\infty)}\right\} \neq 0.$$

Having calculated the matrix $\mathcal{G}^{\pm} = (\mathcal{U}_{\mathbb{R}}^{\pm})^{-1}\mathcal{V}_{\mathbb{R}}^{\pm}$, we obtain $\mathcal{G}^{\pm} = \chi_{\mathbb{R}_-} + \mathcal{A}^{\pm}\chi_{(0,1)} + \mathcal{B}^{\pm}\chi_{(1,\infty)}$, where the matrices $\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ are given by formulas (3.2). The operator $\mathcal{R}(\mathcal{G}^{\pm}) = P_{\mathbb{R}}^{+} + \mathcal{G}^{\pm}P_{\mathbb{R}}^{-}$ is invertible on the space $L_p^2(\mathbb{R}, \varrho)$ if and only if the operator $\mathcal{D}_{\mathbb{R}_+}^{\pm}$ is invertible on the space $L_p^2(\mathbb{R}_+, \varrho)$.

Applying [5, Corollary 2] to the operator $\mathcal{R}(\mathcal{G}^{\pm})$ we complete the proof of the theorem. □

Analogous results can be obtained for the operators $A \in [L_p(\mathbb{T}, \rho)]$ with an orientation-preserving shift $W_{\mathbb{T}} = W_2$, $(W_{\mathbb{T}}\varphi)(t) = \varphi(-t)$:

$$A = aI_{\mathbb{T}} + bW + cS_{\mathbb{T}} + dWS_{\mathbb{T}};$$

as well for the characteristic singular integral operators $D_{\mathbb{T}} \in [L_p^2(\mathbb{T}, \rho)]$:

$$\mathcal{D}_{\mathbb{R}} = \begin{bmatrix} a(x) & b(x) \\ b(-x) & a(-x)\end{bmatrix} I_{\mathbb{R}} + \begin{bmatrix} c(x) & -d(x) \\ d(-x) & -c(-x)\end{bmatrix} S_{\mathbb{R}},$$

where the coefficients of $A$ would be such that the coefficients of the operator $\mathcal{F}^{-1}A\mathcal{F}$ from the operator equality (1.2) would be piecewise constant matrices with three points of discontinuity.

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