PROPAGATION OF ELASTIC WAVES ALONG INTERFACES IN LAYERED BEAMS

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Abstract An asymptotic model is proposed for the analysis of a *long-wave* dynamic model for a layered structure with an imperfect interface. Two layers of isotropic material are connected by a thin and soft adhesive: effectively the layer of adhesive can be described as a surface of discontinuity for the longitudinal displacement. The asymptotic method enables us to derive the *lower-dimensional* differential equations that describe waves associated with the displacement jump across the adhesive.

1. INTRODUCTION

This paper is based on the work [1], [2], [3] on modelling of thin-walled layered structures with high contrast in the elastic properties of the layers. In real physical structures, these models describe adhesive joints. The challenge in the asymptotic analysis is that the problem involves two small parameters: a geometrical parameter characterising the normalised thickness of the beam, and a physical small parameter corresponding to a normalised Young's modulus of the interior adhesive layer. The limit problems depend on the relation between these parameters. The study of the corresponding static problems was presented in [3].

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The new development given here is in the analysis of the wave propagation problem for a layered structure containing an adhesive joint. We shall study discontinuity waves propagating along the adhesive joint. It is appropriate to mention the relevant work [4] and [5] on the vibrational response of plates in vacuo and vibrations of multi-layered beams.

The paper is organised as follows. Sections 2 and 3 describe the geometry and governing equations. Section 4 contains an outline of the structure of the asymptotic expansions. The formal asymptotic algorithm is implemented in Section 5. Section 6 gives an example, which illustrates the lower-dimensional asymptotic model.

2. THE GEOMETRY OF THE SANDWICH BEAM

In this section we define the geometry of a two-dimensional isotropic thin layered structure with an adhesive joint. The formulation of the problem includes two small parameters: the normalised thickness of the structure and the relative stiffness of the adhesive (similar to [1] and [3]).

Let us consider a thin rectangular domain which consists of three layers:

$$\begin{aligned}
\Omega_1 &= \{\mathbf{x} \in \mathbb{R}^2 : |x_1| < l\,,\ \epsilon(h/2 - h_1) + \epsilon^2 h_0 < x_2 < \epsilon h/2 + \epsilon^2 h_0\},\\
\Omega_2 &= \{\mathbf{x} \in \mathbb{R}^2 : |x_1| < l\,,\ -\epsilon h/2 < x_2 < -\epsilon h/2 + \epsilon h_2\},\\
\Omega_0 &= \{\mathbf{x} \in \mathbb{R}^2 : |x_1| < l\,,\ -\epsilon(h/2 - h_2) < x_2 < -\epsilon(h/2 - h_2) + \epsilon^2 h_0\},
\end{aligned}$$

where l and h_i , $i = 0, 1, 2$, have the same order of magnitude. Also we define h as $h = h_1 + h_2$. The elastic materials of the regions Ω_i are characterised by the Youngs moduli E_i and by the values ν_i of the Poisson ratio. The index i throughout the paper takes the values $0, 1$ and 2. By λ_i, μ_i we denote the Lamé constants of the elastic materials which are given as

$$\lambda_i = \frac{E_i \nu_i}{(1+\nu_i)(1-2\nu_i)}, \quad \mu_i = \frac{E_i}{2(1+\nu_i)}. \tag{2.1}$$

The interface boundary includes two parts, S_+ and S_-, specified by

$$\begin{aligned}
S_+ &= \{\mathbf{x} : |x_1| < l\,,\ x_2 = -\epsilon(h/2 - h_2) + \epsilon^2 h_0\},\\
S_- &= \{\mathbf{x} : |x_1| < l\,,\ x_2 = -\epsilon(h/2 - h_2)\}.
\end{aligned} \tag{2.2}$$

The upper and lower surfaces of the compound region are

$$\begin{aligned}
\Gamma_+ &= \{\mathbf{x} : |x_1| < l\,,\ x_2 = \epsilon^2 h_0 + \epsilon h/2\},\\
\Gamma_- &= \{\mathbf{x} : |x_1| < l\,,\ x_2 = -\epsilon h/2\}.
\end{aligned}$$

3. FORMULATION OF THE PROBLEM

In this section we consider propagation of elastic waves and the state of plane strain in the three-layered medium introduced in Section 2. Thus, the displacement field given by $\mathbf{u}^{(i)} = (u_1^{(i)}(\mathbf{x},t), u_2^{(i)}(\mathbf{x},t))$ with $\mathbf{x} = (x_1, x_2)$, satisfies the system

$$\mu_i \nabla^2 \mathbf{u}^{(i)} + (\lambda_i + \mu_i)\nabla\nabla\cdot\mathbf{u}^{(i)} = \rho_i \frac{\partial^2}{\partial t^2}\mathbf{u}^{(i)}, \mathbf{x} \in \Omega_i;\ i = 0, 1, 2. \quad (3.1)$$

Here t denotes the time variable and ρ_i the density of the material at the region Ω_i.

For the surfaces of the compound region Ω_ϵ we prescribe free-traction conditions:

$$\mu_i\left(\frac{\partial u_2^{(i)}}{\partial x_1} + \frac{\partial u_1^{(i)}}{\partial x_2}\right) = 0,\ (2\mu_i + \lambda_i)\frac{\partial u_2^{(i)}}{\partial x_2} + \lambda_i \frac{\partial u_1^{(i)}}{\partial x_1} = 0, \text{ on} \quad (3.2)$$

on Γ_+ $(i = 1)$ and Γ_- $(i = 2)$. On the interface surfaces, the displacement and traction continuity conditions are given by

$$\begin{aligned}
\mu_i\left(\frac{\partial u_2^{(i)}}{\partial x_1} + \frac{\partial u_1^{(i)}}{\partial x_2}\right) &= \mu_0\left(\frac{\partial u_2^{(0)}}{\partial x_1} + \frac{\partial u_1^{(0)}}{\partial x_2}\right), \\
(2\mu_i + \lambda_i)\frac{\partial u_2^{(i)}}{\partial x_2} + \lambda_i\frac{\partial u_1^{(i)}}{\partial x_1} &= (2\mu_0 + \lambda_0)\frac{\partial u_2^{(0)}}{\partial x_2} + \lambda_0\frac{\partial u_1^{(0)}}{\partial x_1}, \\
\mathbf{u}^{(i)} &= \mathbf{u}^{(0)},
\end{aligned} \quad (3.3)$$

on S_+ $(i = 1)$ and S_- $(i = 2)$. We are interested in the analysis of time-harmonic vibrations of the beam and propagation of waves along the thin interface layer.

4. THE STRUCTURE OF THE ASYMPTOTIC EXPANSIONS

In this section, additionally to the analysis given in [6], two time-scales are defined for each component of the displacement vector.

Stretched variables ξ_i, $i = 0, 1, 2$, are introduced to describe the transverse behaviour of the fields across the thickness of the beam and are given as

$$\begin{aligned}
\xi_0 &= \epsilon^{-2}(x_2 + \epsilon(h/2 - h_2) - \epsilon^2 h_0/2), \\
\xi_1 &= \epsilon^{-1}(x_2 - \epsilon^2 h_0 - \epsilon h_2/2), \\
\xi_2 &= \epsilon^{-1}(x_2 + \epsilon h_1/2).
\end{aligned} \quad (4.1)$$

In this way one can verify that

$$\xi_i \in [-h_i/2, h_i/2],\ \ i = 1, 2;\ \ \xi_0 \in [-h_0/2, h_0/2] \quad (4.2)$$

and

$$\partial_{x_2} = \epsilon^{-2}\partial_{\xi_0},\ \ \partial_{x_2} = \epsilon^{-1}\partial_{\xi_i},\ \ i = 1, 2, \quad (4.3)$$

where the notation ∂_α means the partial derivative with respect to α.

Longitudinal vibrations of a thin-walled structure occur at higher frequencies compared to flexural vibrations; slow and fast time variables are used to describe flexural and longitudinal vibrations, respectively.

The displacement field $\mathbf{u}^{(i)}$ is sought in the form of the following asymptotic expansions

$$\mathbf{u}^{(i)} \sim \mathbf{u}^{(i,0)}(x_1, \xi_i, \tau, T) + \epsilon \mathbf{u}^{(i,1)}(x_1, \xi_i, \tau, T) + \epsilon^2 \mathbf{u}^{(i,2)}(x_1, \xi_i, \tau, T) \tag{4.4}$$

where T and τ are scaled variables. Assuming that $T = \epsilon t$, (the slow variable) and $\tau \equiv t$ (the fast variable), we obtain

$$\partial_t^2 = \partial_\tau^2 + 2\epsilon \partial_{\tau T}^2 + \epsilon^2 \partial_T^2. \tag{4.5}$$

The displacement field is split into two terms as follows

$$u_j^{(i)}(x_1, \xi_i, t) = \tilde{u}_j^{(i)}(x_1, \xi_i, \tau) + \bar{u}_j^{(i)}(x_1, \xi_i, T). \tag{4.6}$$

If one substitutes the series (4.4) into (3.1), the boundary conditions (3.2), and analyses the coefficients near like powers of ϵ, it follows that the following recurrence relations hold on the cross-section

$$\mu_i \partial_{\xi_i}^2 u_1^{(i,k)} + (\lambda_i + \mu_i)\partial_{\xi_i x_1}^2 u_2^{(i,k-1)} + (\lambda_i + 2\mu_i)\partial_{x_1}^2 u_1^{(i,k-2)} = \rho_i \left\{ \partial_\tau^2 u_1^{(i,k-2)} + \partial_T^2 u_1^{(i,k-4)} \right\}, \tag{4.7}$$

$$(2\mu_i + \lambda_i)\partial_{\xi_i}^2 u_2^{(i,k)} + (\lambda_i + \mu_i)\partial_{\xi_i x_1}^2 u_1^{(i,k-1)} + \mu_i \partial_{x_1}^2 u_2^{(i,k-2)} = \rho_i \left\{ \partial_\tau^2 u_2^{(i,k-2)} + \partial_T^2 u_2^{(i,k-4)} \right\}, \tag{4.8}$$

for $\Omega_i, i = 1, 2$. Due to the fact that the middle layer is softer than the others, we use the relationship $E_0 = \epsilon^3 E$, where $E \sim E_1 \sim E_2$ (see (2.1)) and obtain

$$\mu \partial_{\xi_0}^2 u_1^{(0,k)} + (\lambda + \mu)\partial_{\xi_0 x_1}^2 u_2^{(0,k-2)} + (\lambda + 2\mu)\partial_{x_1}^2 u_1^{(0,k-4)} = \rho_0 \left\{ \partial_\tau^2 u_1^{(0,k-1)} + \partial_T^2 u_1^{(0,k-3)} \right\}, \tag{4.9}$$

$$(2\mu + \lambda)\partial_{\xi_0}^2 u_2^{(0,k)} + (\lambda + \mu)\partial_{\xi_0 x_1}^2 u_1^{(j,k-2)} + \mu \partial_{x_1}^2 u_2^{(0,k-4)} = \rho_0 \left\{ \partial_\tau^2 u_2^{(0,k-1)} + \partial_T^2 u_2^{(0,k-3)} \right\}, \tag{4.10}$$

in Ω_0. As for the static case (see [1]), we have the following interface boundary conditions

$$\begin{aligned} \mu_i(\partial_{\xi_1} u_1^{(i,k)} + \partial_{x_1} u_2^{(i,k-1)}) &= \mu(\partial_{\xi_0} u_1^{(0,k-2)} + \partial_{x_1} u_2^{(0,k-4)}), \\ (2\mu_i + \lambda_i)\partial_{\xi_1} u_2^{(i,k)} + \lambda_i \partial_{x_1} u_1^{(i,k-1)} &= (2\mu + \lambda)\partial_{\xi_0} u_2^{(0,k-2)} + \lambda \partial_{x_1} u_1^{(0,k-4)}, \\ u_j^{(0,k)} &= u_j^{(i,k)} \quad j = 1, 2, \end{aligned} \tag{4.11}$$

on S_+ $(i = 1)$ and S_- $(i = 2)$.

For the upper and lower surfaces we have

$$\begin{aligned} \mu_i(\partial_{\xi_1} u_1^{(i,k)} + \partial_{x_1} u_2^{(i,k-1)}) &= 0, \\ (2\mu_i + \lambda_i)\partial_{\xi_1} u_2^{(i,k)} + \lambda_i \partial_{x_1} u_1^{(i,k-1)} &= 0, \end{aligned} \tag{4.12}$$

on Γ_+ $(i = 1)$ and Γ_- $(i = 2)$.

5. FORMAL ASYMPTOTIC ALGORITHM

At each step of the asymptotic algorithm, solvability conditions of the model boundary value problems (BVP) on the cross-section are formulated and analysed.

For the transverse components, the following condition for the *slow components* is obtained

$$\boxed{\bar{u}_2^{(1,0)} = \bar{u}_2^{(2,0)} = \bar{u}_2^{(0,0)} \equiv \bar{u}_2^{(0)}.} \tag{5.1}$$

This means that to the leading-order, all points on the cross-section of the beam have the same transverse displacement in slow motions. This agrees with the Kirchhoff hypothesis adopted in the classical theory of flexural motions of elastic beams. For the *fast components* the solvability conditions of relevant model problems give a system of ordinary differential equations

$$\boxed{\frac{\rho_1 h_0 h_1}{\lambda + 2\mu} \partial_\tau^2 \tilde{u}_2^{(1,0)} + \tilde{u}_2^{(1,0)} - \tilde{u}_2^{(2,0)} = 0,} \tag{5.2}$$

$$\boxed{\frac{\rho_2 h_0 h_2}{\lambda + 2\mu} \partial_\tau^2 \tilde{u}_2^{(2,0)} + \tilde{u}_2^{(2,0)} - \tilde{u}_2^{(1,0)} = 0.} \tag{5.3}$$

These equations describe the transverse x_1-independent motion within a composite beam. With these conditions taken into account, the functions $u_2^{(i,2)}$ are given by

$$\begin{aligned} u_2^{(1,2)} = {} & \frac{\lambda_1}{2\mu_1 + \lambda_1} \left[\frac{\xi_1^2}{2} (\partial_{x_1}^2 \tilde{u}_2^{(1,0)} + \partial_{x_1}^2 \bar{u}_2^{(1,0)}) - \xi_1 (\partial_{x_1} \tilde{v}^{(1)} + \partial_{x_1} \bar{v}^{(1)}) \right] \\ & + \frac{\rho_1}{2\mu_1 + \lambda_1} \left[\frac{\xi_1^2}{2} - \frac{\xi_1 h_1}{2} \right] \partial_\tau^2 \tilde{u}_2^{(1,0)}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} u_2^{(2,2)} = {} & \frac{\lambda_2}{2\mu_2 + \lambda_2} \left[\frac{\xi_2^2}{2} (\partial_{x_1}^2 \tilde{u}_2^{(2,0)} + \partial_{x_1}^2 \bar{u}_2^{(2,0)}) - \xi_2 (\partial_{x_1} \tilde{v}^{(2)} + \partial_{x_1} \bar{v}^{(2)}) \right] \\ & + \frac{\rho_2}{2\mu_2 + \lambda_2} \left[\frac{\xi_2^2}{2} + \frac{\xi_2 h_2}{2} \right] \partial_\tau^2 \tilde{u}_2^{(2,0)}. \end{aligned} \tag{5.5}$$

We note that

$$u_1^{(i,1)} = -\xi_i \left[\partial_{x_1}\tilde{u}_2^{(i,0)} + \partial_{x_1}\bar{u}_2^{(i,0)}\right] + \tilde{v}^{(i)}(x_1,\tau) + \bar{v}^{(i)}(x_1,T),\ i = 1,2$$

The functions $\bar{v}^{(i)}$ satisfy second-order differential equations derived as solvability conditions (when $k = 3$) for model problems associated with "slow" motions.

$$\boxed{\frac{4(\lambda_1+\mu_1)}{2\mu_1+\lambda_1}h_1\partial^2_{x_1}\bar{v}^{(1)} = \frac{\mu}{\mu_1 h_0}\left\{\frac{h_1+h_2}{2}\partial_{x_1}\bar{u}_2^{(0)} + \bar{v}^{(1)} - \bar{v}^{(2)}\right\},} \tag{5.6}$$

$$\boxed{\frac{4(\lambda_2+\mu_2)}{2\mu_2+\lambda_2}h_2\partial^2_{x_1}\bar{v}^{(2)} = -\frac{\mu}{\mu_2 h_0}\left\{\frac{h_1+h_2}{2}\partial_{x_1}\bar{u}_2^{(0)} + \bar{v}^{(1)} - \bar{v}^{(2)}\right\}.} \tag{5.7}$$

For the fast motions we obtain

$$\boxed{\begin{aligned}&\frac{4(\lambda_1+\mu_1)}{2\mu_1+\lambda_1}h_1\partial^2_{x_1}\tilde{v}^{(1)} + \frac{\rho_1 h_1}{\mu_1}\left\{\frac{\lambda_1 h_1}{2(2\mu_1+\lambda_1)}\partial^3_{x_1\tau^2}\tilde{u}_2^{(1,0)} - \partial^2_\tau\tilde{v}^{(1)}\right\}\\ &= \frac{\mu}{\mu_1 h_0}\left\{\frac{h_1}{2}\partial_{x_1}\tilde{u}_2^{(1,0)} + \frac{h_2}{2}\partial_{x_1}\tilde{u}_2^{(2,0)} + \tilde{v}^{(1)} - \tilde{v}^{(2)}\right\},\end{aligned}} \tag{5.8}$$

$$\boxed{\begin{aligned}&\frac{4(\lambda_2+\mu_2)}{2\mu_2+\lambda_2}h_2\partial^2_{x_1}\tilde{v}^{(2)} + \frac{\rho_2 h_2}{\mu_2}\left\{\frac{\lambda_2 h_2}{2(2\mu_2+\lambda_2)}\partial^3_{x_1\tau^2}\tilde{u}_2^{(2,0)} - \partial^2_\tau\tilde{v}^{(2)}\right\}\\ &= -\frac{\mu}{\mu_2 h_0}\left\{\frac{h_1}{2}\partial_{x_1}\tilde{u}_2^{(1,0)} + \frac{h_2}{2}\partial_{x_1}\tilde{u}_2^{(2,0)} + \bar{v}^{(1)} - \bar{v}^{(2)}\right\}.\end{aligned}} \tag{5.9}$$

Taking into account solvability conditions for the Neumann BVP on the cross-section at the step $k = 4$ for the transverse displacement components we can establish the following equations:

$$\boxed{\begin{aligned}&\frac{1}{3}\frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1}h_1^3\partial^4_{x_1}\bar{u}_2^{(0)} - 2\mu_1\frac{\lambda_1+\mu_1}{2\mu_1+\lambda_1}h_1^2\partial^3_{x_1}\bar{v}^{(1)} + \rho_1 h_1\partial^2_T\bar{u}_2^{(0)}\\ &+\frac{2\mu+\lambda}{8h_0^2}\left\{\frac{\lambda_1 h_1}{2h_0(2\mu_1+\lambda_1)}\partial_{x_1}\bar{v}^{(1)} + \frac{\lambda_2 h_2}{2h_0(2\mu_2+\lambda_2)}\partial_{x_1}\bar{v}^{(2)}\right.\\ &\left.\left[\frac{\lambda_1 h_1 h_0}{2\mu_1+\lambda_1} - \frac{\lambda_2 h_2 h_0}{2\mu_2+\lambda_2}\right]\partial^2_{x_1}\bar{u}_2^{(0)}\right\} = 0,\end{aligned}} \tag{5.10}$$

$$\boxed{\begin{aligned}&\frac{1}{3}\frac{\mu_2(\mu_2+\lambda_2)}{2\mu_2+\lambda_2}h_2^3\partial^4_{x_1}\bar{u}_2^{(0)} + 2\mu_2\frac{\lambda_2+\mu_2}{2\mu_2+\lambda_2}h_2^2\partial^3_{x_1}\bar{v}^{(2)} + \rho_2 h_2\partial^2_T\bar{u}_2^{(0)}\\ &-\frac{2\mu+\lambda}{8h_0^2}\left\{\frac{\lambda_1 h_1}{2h_0(2\mu_1+\lambda_1)}\partial_{x_1}\bar{v}^{(1)} + \frac{\lambda_2 h_2}{2h_0(2\mu_2+\lambda_2)}\partial_{x_1}\bar{v}^{(2)}\right.\\ &\left.\left[\frac{\lambda_1 h_1 h_0}{2\mu_1+\lambda_1} - \frac{\lambda_2 h_2 h_0}{2\mu_2+\lambda_2}\right]\partial^2_{x_1}\bar{u}_2^{(0)}\right\} = 0.\end{aligned}} \tag{5.11}$$

We remark that these equations do not involve *fast* functions $\tilde{u}_2^{(i,4)}, i = 1, 2$.

6. ILLUSTRATIVE EXAMPLE AND CONCLUDING REMARKS

As shown in the previous section, the asymptotic algorithm allows one to find explicitly lower-dimensional differential equations describing longitudinal and flexural vibrations within a composite beam.

Slow motions occur in accordance with the equations (5.6),(5.7),(5.10) and (5.11). The equations for the transverse components involve the fourth-order derivative in x_1, which is consistent with classical results of the theory of elastic beams (see, for example, [8]). However the presence of an imperfect interface provides a coupling between the longitudinal and transverse displacements associated with a slow motion.

Fast motions are described by the second-order differential equations (5.2), (5.3), (5.8) and (5.9). These motions may involve a longitudinal displacement jump, and discontinuity waves might propagate along the soft interface. Since the transverse fast motions occur according to equations (5.2) and (5.3), which do not include derivatives with respect to x_1, the transverse vibrations do not generate waves propagating along the adhesive joint.

Next, we consider an illustrative example. Assume that the upper and lower layers have the same thickness $h_1 = h_2$ and made of the same material ($\mu_1 = \mu_2, \lambda_1 = \lambda_2, \rho_1 = \rho_2$). Combining equations (5.10) and (5.11), and using equations (5.6) and (5.7), we obtain

$$\boxed{\frac{h_1^3}{3}\frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1}\partial_{x_1}^4\bar{u}_2^{(0)} + \rho_1 h_1 \partial_T^2 \bar{u}_2^{(0)} = 0.} \tag{6.1}$$

Seeking a solution of this equation in the form

$$\bar{u}_2^{(0)} = A\exp\left(ikx_1 - i\Omega T\right)$$

we derive the corresponding characteristic equation

$$\frac{2h_1^3}{3}\frac{\mu_1(\mu_1+\lambda_1)}{2\mu_1+\lambda_1}k^4 - 2\rho_1 h_1 \Omega^2 = 0.$$

The previous equation reduces to the standard dispersion relation attributed to the Kirchhoff theory:

$$Dk^4 - \rho_l h_1 \Omega^2 = 0, \tag{6.2}$$

where $D = \frac{Eh_1^3}{12(1-\nu_1^2)}$.

For the case of fast motions we obtain the following system of differential equations:

$$
\begin{aligned}
&\frac{4(\lambda_1+\mu_1)}{2\mu_1+\lambda_1} h_1 \partial^2_{x_1} \tilde{v}^{(1)} + \frac{\rho_1 h_1}{\mu_1}\left\{ \frac{\lambda_1 h_1}{2(2\mu_1+\lambda_1)} \partial^3_{x_1\tau^2} \tilde{u}_2^{(1,0)} - \partial^2_\tau \tilde{v}^{(1)} \right\} \\
&- \frac{\mu}{\mu_1 h_0}\left\{ \frac{h_1}{2}\partial_{x_1}\tilde{u}_2^{(1,0)} + \frac{h_1}{2}\partial_{x_1}\tilde{u}_2^{(2,0)} + \tilde{v}^{(1)} - \tilde{v}^{(2)} \right\} = 0, \qquad (6.3) \\
&\frac{4(\lambda_1+\mu_1)}{2\mu_1+\lambda_1} h_1 \partial^2_{x_1} \tilde{v}^{(2)} + \frac{\rho_1 h_1}{\mu_1}\left\{ \frac{\lambda_1 h_1}{2(2\mu_1+\lambda_1)} \partial^3_{x_1\tau^2} \tilde{u}_2^{(2,0)} - \partial^2_\tau \tilde{v}^{(2)} \right\} \\
&+ \frac{\mu}{\mu_1 h_0}\left\{ \frac{h_1}{2}\partial_{x_1}\tilde{u}_2^{(1,0)} + \frac{h_1}{2}\partial_{x_1}\tilde{u}_2^{(2,0)} + \bar{v}^{(1)} - \bar{v}^{(2)} \right\} = 0, \qquad (6.4) \\
&\frac{\rho_1 h_0 h_1}{2\mu+\lambda}\partial^2_\tau \tilde{u}_2^{(1,0)} + \tilde{u}_2^{(1,0)} - \tilde{u}_2^{(2,0)} = 0, \qquad (6.5) \\
&\frac{\rho_1 h_0 h_1}{2\mu+\lambda}\partial^2_\tau \tilde{u}_2^{(2,0)} + \tilde{u}_2^{(2,0)} - \tilde{u}_2^{(1,0)} = 0. \qquad (6.6)
\end{aligned}
$$

A solution for the homogeneous problem (6.3)–(6.6) is sought in the form

$$
\begin{aligned}
\tilde{v}^{(j)} &= A_j \exp\left(ikx_1 - i\omega\tau\right), \\
\tilde{u}_2^{(j,0)} &= B_j \exp\left(ikx_1 - i\omega\tau\right),
\end{aligned}
$$

where $j = 1, 2$. The corresponding characteristic equation has the roots given by

$$\omega_1^2 = 0, \qquad (6.7)$$

$$\omega_2^2 = \frac{2(2\mu+\lambda)}{\rho_1 h_1 h_0}. \qquad (6.8)$$

The first root (6.7) is related to a uniform transverse displacement of all three layers, and the second root (6.8) corresponds to an anti-phase vibration of the upper and lower layers, relative to each other.

The equation

$$\omega^2 = \frac{4\mu_1(\mu_1+\lambda_1)}{\rho_1(2\mu_1+\lambda_1)}k^2, \qquad (6.9)$$

corresponds to uniform longitudinal motions of the whole layered structure, with no displacement jump across the adhesive layer.

The equation

$$\omega^2 = \frac{4\mu_1(\mu_1+\lambda_1)}{\rho_1(2\mu_1+\lambda_1)}k^2 + \frac{2\mu}{\rho_1 h_1 h_0} \qquad (6.10)$$

describes anti-phase longitudinal motions of the upper and lower layers, representing shear mode of motions.

We can see that there exists a cut-off frequency which is given by

$$\omega = \omega_3 = \sqrt{\frac{2\mu}{\rho_1 h_1 h_0}}. \tag{6.11}$$

If the frequency of the signal does not exceed the critical value ω_3, the displacement jump may not propagate along the imperfect interface. Let

$$k_1 = \frac{\mu_1 + \lambda_1}{\sqrt{2h_1 h_0 \mu_1(\mu_1 + \lambda_1)}}, \tag{6.12}$$

$$k_2 = \frac{2\mu_1 + \lambda_1}{\sqrt{2h_1 h_0 \mu_1(\mu_1 + \lambda_1)}}. \tag{6.13}$$

The intersection points of the dispersion curves given by (k_1, ω_2) and (k_2, ω_2), correspond to the resonance modes involving transverse and longitudinal vibrations.

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