CONTINUOUS AND DISCRETE FLOWS ON OPERATOR ALGEBRAS

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Abstract. Let $(N, \mathbb{R}, \theta)$ be a centrally ergodic W* dynamical system. When N is not a factor, we show that, for each $t \neq 0$, the crossed product induced by the time t automorphism θ_t is not a factor if and only if there exist a rational number r and an eigenvalue s of the restriction of θ to the center of N, such that $rst = 2\pi$. In the C* setting, minimality seems to be the notion corresponding to central ergodicity. We show that if $(A, \mathbb{R}, \alpha)$ is a minimal unital C* dynamical system and A is either prime or commutative but not simple, then, for each $t \neq 0$, the crossed product induced by the time t automorphism α_t is not simple if and only if there exist a rational number r and an eigenvalue s of the restriction of α to the center of A, such that $rst = 2\pi$.

Introduction

Recall that a flow (Y, T) is a pair consisting of a compact metric space Y and an action $T: Y \times \mathbb{R} \to Y$. The time t map of the flow (Y, T) is the automorphism $T^t: Y \to Y$. We say that a flow (Y, T) is minimal if there is no nontrivial closed invariant subspaces of Y.

If (Y, T) is a minimal flow then, for $t \neq 0$, Proposition 1.5 in [2] shows that the crossed product induced by the time $1/t$ map $T^{\frac{1}{t}}$ is not minimal if and only if there exists a rational number r and an eigenvalue s of T such that $rst = 2\pi$. (The 2π term appears in this equality when we remove it from the definition of eigenvalue in [2, Definition 1.1].) In the noncommutative setting, a flow is a C* dynamical system $(A, \mathbb{R}, \alpha)$ consisting of a (unital) C*-algebra A and a one-parameter group of automorphisms $\alpha: \mathbb{R} \to \mathrm{Aut}(A)$. The action α is said to be *minimal* (or, equivalently, we say that A is α-simple) if A has no nontrivial invariant ideals. When $A = C(Y)$ is a commutative (unital) C*-algebra, a flow α on A is induced by a flow T on Y, that is, $\alpha_t(f) = f \circ T^t$ for all f in A and for all t in $\mathbb{R}$. Then α is minimal if and only if T is minimal in the classical sense.

The aim in this paper is to extend some of the results in [2] to the noncommutative case, in other words, given a minimal C*-dynamical system $(A, \mathbb{R}, \alpha)$, we try to relate the values of t for which the crossed product induced by the time t automorphism α_t is not simple, with the eigenvalues of (a restriction of) α. We are unable to answer this question in general. Our partial results are contained in Section 2 of this paper. It is natural to ask what is the corresponding problem for von Neumann algebras. It turns out that minimality in the C*-algebra setting corresponds to central ergodicity in the W*-algebra setting (recall that a W* dynamical system $(N, \mathbb{R}, \theta)$ is said to be *centrally ergodic* if the restriction of θ to the center of N is ergodic). Indeed, if $(A, \mathbb{R}, \alpha)$ is a minimal C* dynamical system then the crossed product $A \rtimes_\alpha \mathbb{R}$ is minimal if and only if the strong Connes spectrum of α is $\mathbb{R}$, cf. [3]; whilst if $(N, \mathbb{R}, \theta)$ is a centrally ergodic W* dynamical system then the crossed product $N \rtimes_\theta \mathbb{R}$ is a factor if and only if the Connes spectrum of θ is $\mathbb{R}$, cf. [11, Corollay XI.2.8, pg 336]. In Section 1 we discuss this version of the problem, which we are able to solve satisfactorily. We conclude each of the sections with some open problems.

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1. W* dynamical systems

Let $(N, \mathbb{R}, \theta)$ be a W* dynamical system. We say that a real number s is an eigenvalue for θ if there exists a nonzero a in N such that, for all t in $\mathbb{R}$, we have $\theta_t(a) = e^{ist}a$. When θ is ergodic (that is, the set of fixed points of θ is $\mathbb{C}1$), the set of eigenvalues, which we denote by $\Lambda(\theta)$, is a subgroup of $\mathbb{R}$, cf. [8].

Our first lemma is known for the case when the underlying measurable space of the center of the von Neumann algebra in question is a probability space, see [1, Lemma 12.1.1, pg 326]. The proof there, however, can not be adapted to a more general situation. Our proof here is essentially contained in the proof of [10, Theorem 10.6].

Lemma 1.1. *Let $(N, \mathbb{R}, \theta)$ be a centrally ergodic W* dynamical system and let T be a strictly positive real number. If θ_T is not centrally ergodic then there exists a nonzero eigenvalue s of the restriction of θ to the center of N, such that $e^{isT} = 1$.*

Proof. Suppose that θ_T is not centrally ergodic. Then $\mathcal{A} = Z(N)^{\theta_T}$ is a commutative von Newmann algebra which is not reduced to the scalars. Furthermore, $\mathcal{A}$ is θ-invariant and the action of θ on $\mathcal{A}$ is ergodic and periodic. Hence the action of θ on $\mathcal{A}$ is transitive. Thus there exists $T_0 > 0$ such that the action of θ on $\mathcal{A}$ is isomorphic to the canonical action of $\mathbb{R}$ on $L^\infty(\mathbb{R}/T_0\mathbb{Z})$. Since $s = \frac{2\pi}{T_0}$ is an eigenvalue for this action (with eigenfunction defined by $f(t + T_0\mathbb{Z}) = e^{ist}$, for all t in $\mathbb{R}$), then we have that s is an eigenvalue of the action of θ on $\mathcal{A}$, and so, s is an eigenvalue of the action of θ on $Z(N)$. Since the action of θ on $\mathcal{A}$ is periodic with period T_0, there is k in $\mathbb{Z}$ such that $T = kT_0$. Hence $s = \frac{2\pi}{T_0} = \frac{2\pi k}{T}$ is a nonzero eigenvalue for the restriction of θ to $Z(N)$ and $e^{isT} = e^{i2\pi k} = 1$, as wanted. $\square$

The next result can be regarded as the W* version of [2, Proposition 1.5].

Proposition 1.2. *Let $(N, \mathbb{R}, \theta)$ be a centrally ergodic W* dynamical system and let $\widetilde{\theta}$ denote the restriction of θ to the center of N. Consider the map with domain $\frac{1}{2\pi}\mathbb{Q} \otimes \Lambda\left(\widetilde{\theta}\right)$ and codomain $\mathbb{R}$ defined by $\frac{r}{2\pi} \otimes s \mapsto \frac{rs}{2\pi}$. This map is a $\mathbb{Q}$-linear monomorphism with range equal to*

$$\left\{0\right\} \cup \left\{t \in \mathbb{R} \setminus \{0\} : \theta_{\frac{1}{t}} \text{ is not centrally ergodic}\right\}.$$

Hence the set above is a $\mathbb{Q}$-linear subspace of $\mathbb{R}$ isomorphic to $\frac{1}{2\pi}\mathbb{Q} \otimes \Lambda\left(\widetilde{\theta}\right)$.

Proof. It is clear that the map $\frac{r}{2\pi} \otimes s \mapsto \frac{rs}{2\pi}$ is a $\mathbb{Q}$-linear monomorphism. We only need to show that the range is as proposed. Let $r = \frac{p}{q}$ be a nonzero rational number and let s be a nonzero eigenvalue for $\widetilde{\theta}$. Put $T = \frac{rs}{2\pi} = \frac{ps}{2\pi q}$. We show that $(N, \mathbb{Z}, \theta_{\frac{1}{T}})$ is not centrally ergodic. Since $\Lambda\left(\widetilde{\theta}\right)$ is a group, we get that ps is also an eigenvalue. Therefore, there exists a nonzero a in the center of N such that $\theta_t(a) = e^{ipst}a$, for all $t \in \mathbb{R}$. Notice that a is not a scalar since $ps \neq 0$. Hence

$$\theta_{\frac{1}{T}}(a) = \theta_{\frac{2\pi q}{ps}}(a) = e^{i2\pi q}a = a.$$

Thus $(N, \mathbb{Z}, \theta_{\frac{1}{T}})$ is not centrally ergodic. Conversely, assume that $(N, \mathbb{Z}, \theta_{\frac{1}{T}})$ is not centrally ergodic for some $T \neq 0$. By Lemma 1.1, there is $0 \neq s \in \Lambda\left(\widetilde{\theta}\right)$ such that $e^{is\frac{1}{T}} = 1$. Therefore $s\frac{1}{T} = 2\pi k$ for some $k \in \mathbb{Z}$ and so $T = \frac{1}{2\pi k}s$, as desired. □

We now prove the following easy lemma.

Lemma 1.3. *An inner automorphism on a von Neumann algebra N is centrally ergodic if and only if N is a factor.*

Proof. Let θ be an inner automorphism on a von Neumann algebra N. If N is a factor then it is clear that θ is centrally ergodic. To prove the converse, suppose that θ is centrally ergodic. Let a be an element in the center of N. Since θ is inner, it follows that $\theta(a) = a$. As θ is centrally ergodic, we conclude that a is a scalar multiple of the identity, as wanted. □

We are ready to prove the main result of this section.

Theorem 1.4. *Let $(N, \mathbb{R}, \theta)$ be a centrally ergodic W^* dynamical system where N is not a factor. Let t be a nonzero real number. Denote by $\widetilde{\theta}$ the restriction of θ to the center of N. The following statements are equivalent.*

(1) *The W^* dynamical system $(N, \mathbb{Z}, \theta_t)$ is not centrally ergodic.*
(2) *The von Neumann algebra $N \rtimes_{\theta_t} \mathbb{Z}$ is not a factor.*
(3) *There exists (r, s) in $\mathbb{Q} \times \Lambda\left(\widetilde{\theta}\right)$ such that $rst = 2\pi$.*

Proof. (1)⇒(2): This is well known, see e.g. [11, Corollay XI.2.8, pg 336] or [7, Theorem 8.11.15, pg 362].

(2)⇒(3): If $N \rtimes_{\theta_t} \mathbb{Z}$ is not a factor then either $(N, \mathbb{Z}, \theta_t)$ is not centrally ergodic or $\Gamma(\theta_t) \neq \mathbb{T}$, cf. [7, Theorem 8.11.15, pg 362]. If $(N, \mathbb{Z}, \theta_t)$ is not centrally ergodic, we use Proposition 1.2 to conclude that $\frac{1}{t} = \frac{r}{2\pi}s$ for some (r, s) in $\mathbb{Q} \times \Lambda\left(\widetilde{\theta}\right)$. Hence $rst = 2\pi$, and we are done. Else, assume that $(N, \mathbb{Z}, \theta_t)$ is centrally ergodic and $\Gamma(\theta_t) \neq \mathbb{T}$. Then $\Gamma(\theta_t)^{\perp} \neq \{0\}$. As $\mathbb{T}$ is compact, we may use [11, Theorem XI.2.9(ii), pg 336] to conclude that $\theta_t^n = \theta_{nt}$ is inner for some nonzero integer n in $\Gamma(\theta_t)^{\perp} \subset \mathbb{Z}$. Since N is not a factor, we get that θ_{nt} is not centrally ergodic, cf. Lemma 1.3. Another application of Proposition 1.2 completes the proof.

(3)⇒(1): Suppose there exists (r, s) in $\mathbb{Q} \times \Lambda\left(\widetilde{\theta}\right)$ such that $rst = 2\pi$. Then $\frac{1}{t} = \frac{r}{2\pi}s$ and so, by Proposition 1.2, the W* dynamical system $(N, \mathbb{Z}, \theta_t)$ is not centrally ergodic. □

Recall that a von Neumann algebra M of type III induces a (unique, up to conjugacy) flow on a von Neumann algebra of type II_∞, which is called the *non-commutative flow of weights of M* or the *associated covariant system of M*, cf. [11, Definition XII.1.3, pg 368]. As an example, we specialize Theorem 1.4 to the type III_λ case, $0 < \lambda < 1$, to get the following.

Corollary 1.5. *Let M be a factor of type* III_λ, $0 < \lambda < 1$, *with associated covariant system* $(N, \mathbb{R}, \theta)$. *Let t be a nonzero real number. The following statements are equivalent.*

1. *The W* dynamical system* $(N, \mathbb{Z}, \theta_t)$ *is not centrally ergodic.*
2. *The von Neumann algebra* $N \rtimes_{\theta_t} \mathbb{Z}$ *is not a factor.*
3. *There exists a rational number r such that* $t = r \log \lambda$.

Proof. Let $\widetilde{\theta}$ denote the restriction of θ to the center of N. In this case $\frac{2\pi}{\log \lambda}\mathbb{Z} = T(M) = \Lambda\left(\widetilde{\theta}\right)$, see [11, Theorem XII.1.6, pg 369] or [9, 28.11, pg 425]. Therefore, using Proposition 1.2, if t is a nonzero real number then $(N, \mathbb{Z}, \theta_t)$ is not centrally ergodic if and only if there is r' in $\mathbb{Q}$ and n in $\mathbb{Z}$ such that $\frac{1}{t} = \frac{r'}{2\pi}\frac{2\pi n}{\log \lambda}$ if and only if $t = r \log \lambda$, where $r = \frac{1}{r' n}$ is rational. $\square$

Let $(N, \mathbb{R}, \theta)$ be a centrally ergodic W* dynamical system. It could be the case that, for all nonzero t, the crossed product induced by the time t map θ_t is a factor: for example, if $(N, \mathbb{R}, \theta)$ is the associated covariant system of a factor of type III_0 (because $\Lambda\left(\widetilde{\theta}\right) = \{0\}$, see [11, Theorem XII.1.6, pg 369] or [9, 28.11, pg 425]). On the other hand, it could be the case that, for every real number t, the crossed product induced by the time t map θ_t is not a factor: for example, if $N \rtimes_\theta \mathbb{R}$ is semisimple, where N is a properly infinite semifinite von Neumann algebra which admits a faithful semifinite normal trace τ such that $\tau \circ \theta_t = e^{-t}\tau$, for all t in $\mathbb{R}$ (because $\Lambda\left(\widetilde{\theta}\right) = \mathbb{R}$, cf. [10, Theorem 8.6]).

Theorem 1.4 is no longer valid when N is a factor. The author is grateful to Professor A. Kishimoto for communicating to us one of his unpublished examples. We adapt his example here to the W* setting. Let N be the von Neumann algebra generated by two unitary operators u and v satisfying $uv = e^{2\pi i s} vu$, where s is an irrational number. Let θ be the flow on N defined by $\theta_t(u) = e^{2\pi i t}u$ and $\theta_t(v) = e^{2\pi i s t}v$, for all t in $\mathbb{R}$. Then θ is centrally ergodic and $\Lambda\left(\widetilde{\theta}\right) = \{0\}$ (because N is a factor). However, the crossed product $N \rtimes_{\theta_1} \mathbb{Z}$ induced by the time 1 automorphism θ_1 is not a factor (because θ_1 is inner). It is worth mentioning that, in fact, the von Neumann algebra $N \rtimes_\theta \mathbb{R}$ is a factor. Therefore, this example satisfies a stronger condition than the one required in Theorem 1.4.

One may ask if a similar result to Proposition 1.2 is true if we substitute the centrally ergodic condition by ergodicity. The example above suggests we should substitute the invariant $\Lambda\left(\widetilde{\theta}\right)$ by $\Lambda(\theta)$. We remark that, by results of Sørmer [8], if θ is a flow on N which has kernel different from $\{0\}$ (or, equivalently, $\mathrm{Sp}(\alpha) \neq \mathbb{R}$, cf. [8, Theorem 3.2]), then N must be abelian [8, Theorem 3.5]. This case, of course, is covered in Proposition 1.2. We conclude this section with some open questions.

Problem 1: Suppose that $(N, \mathbb{R}, \theta)$ is a centrally ergodic flow. If N is a factor, characterize the values of t for which the crossed product associated to the time t automorphism θ_t is a factor.

Problem 2: Suppose that $(N, \mathbb{R}, \theta)$ is an ergodic flow. Characterize the values of t for which the time t automorphism θ_t is ergodic.

2. C* dynamical systems

Let $(A, \mathbb{R}, \alpha)$ be a unital C* dynamical system. We say that a real number s is an eigenvalue for α if there exists a nonzero a in A such that, for all t in $\mathbb{R}$, it follows that $\alpha_t(a) = e^{ist}a$. We denote by $\Lambda(\alpha)$ the set of eigenvalues of α. In this section, we only consider eigenvalues of flows on commutative C*-algebras. If $A = C(Y)$ is commutative and α is induced by a flow T on Y, one may check that the eigenvalues for α are the same as the eigenvalues of T in the classical sense. (We remark that in [2, Definition 1.1] and some places in the literature, a 2π term appears in the classical definition of eigenvalue. We compensate for such constant in the results below.)

Using results by Olesen and Pedersen, we obtain the following.

Proposition 2.1. *Let $(A, \mathbb{R}, \alpha)$ be a C* dynamical system. Assume that for all s in $\mathbb{R}$ and for all nonzero ideal I of A, $I \cap \alpha_s(I) \neq \{0\}$. If α is minimal then, for all nonzero real number t, the automorphism α_t is minimal.*

Proof. Let t be a nonzero real number. If α_t is not minimal then there exists a nontrivial α_t-invariant ideal J of A. Hence J is invariant under $G_0 = t\mathbb{Z}$. Since $\mathbb{R}/G_0$ is compact, we may use [5, Proposition 2.2] to conclude that J contains a nonzero α-invariant ideal I, and so, the ideal I is nontrivial because it is contained in the nontrivial ideal J. This contradicts the minimality of α and completes the proof. □

We remark that if A is prime then the hypothesis of the proposition is satisfied. Also, this hypothesis is equivalent to say that the dual action $\widehat{\alpha}$ has full Connes spectrum, cf. [4, Lemma 3.2].

We can write the following result in a remarkable similarity to Proposition 1.2.

Proposition 2.2. *Let $(A, \mathbb{R}, \alpha)$ be a unital and minimal C* dynamical system and let $\widetilde{\alpha}$ denote the restriction of α to the center of A. Assume that A is either commutative or prime. Consider the map with domain $\frac{1}{2\pi}\mathbb{Q} \otimes \Lambda(\widetilde{\alpha})$ and codomain $\mathbb{R}$ given by $\frac{r}{2\pi} \otimes s \mapsto \frac{rs}{2\pi}$. This map is a $\mathbb{Q}$-linear monomorphism with range equal to*

$$\Big\{0\Big\} \cup \Big\{t \in \mathbb{R} \setminus \{0\} \colon \alpha_{\frac{1}{t}} \text{ is not minimal}\Big\}.$$

Hence the set above is a $\mathbb{Q}$-linear subspace of $\mathbb{R}$ isomorphic to $\frac{1}{2\pi}\mathbb{Q} \otimes \Lambda(\widetilde{\alpha})$.

Proof. If A is commutative, this is [2, Proposition 1.5]. For the case that A is prime, we have that $Z(A) = \mathbb{C}1$ and so $\Lambda(\widetilde{\alpha}) = \{0\}$. This, together with Proposition 2.1, complete the proof. □

We now prove a statement analogous to Lemma 1.3.

Lemma 2.3. *An inner automorphism on a C*-algebra A is minimal if and only if A is simple.*

Proof. Let α be an inner automorphism on a C*-algebra A. If A is simple then it is clear that α is minimal. To prove the converse, suppose that α is minimal. Let I be an ideal of A. Then I is α-invariant because α is inner. Thus I is trivial because α is minimal. Hence A is simple. □

The following is the main result of this section.

Theorem 2.4. *Let $(A,\mathbb{R},\alpha)$ be a unital and minimal C^* dynamical system where A is not simple. Assume that A is either commutative or prime. Let t be a nonzero real number. Denote by $\widetilde{\alpha}$ the restriction of α to the center of A. The following statements are equivalent.*

(1) *The C^* dynamical system $(A,\mathbb{Z},\alpha_t)$ is not minimal.*
(2) *The C^*-algebra $A \rtimes_{\alpha_t} \mathbb{Z}$ is not simple.*
(3) *There exists (r,s) in $\mathbb{Q}\times\Lambda(\widetilde{\alpha})$ such that $rst = 2\pi$.*

Proof. (1)⇒(2): This is well known, see eg. [4, Theorem 6.5] or [3, Theorem 3.5 and Proposition 3.8].

(2)⇒(3): If $A \rtimes_{\alpha_t} \mathbb{Z}$ is not simple then either α_t is not minimal or $\Gamma(\alpha_t) \neq \mathbb{T}$, cf. [4, Theorem 6.5]. If α is not minimal then, using Proposition 2.2, we can find a rational number r and an eigenvalue s in $\Lambda(\widetilde{\alpha})$ such that $\frac{1}{t} = \frac{r}{2\pi}s$. Hence $rst = 2\pi$, as desired. Else, assume that α_t is minimal and $\Gamma(\alpha_t) \neq \mathbb{T}$. Then $\Gamma(\alpha_t)^{\perp} \neq \{0\}$. As $\mathbb{T}$ is compact, we may use [6, Theorem 4.5] to find a nonzero n in $\Gamma(\alpha_t)^{\perp} \subset \mathbb{Z}$ such that $\alpha_t^n = \alpha_{nt}$ is inner. Since A is not simple, we get that α_{nt} is not minimal, cf. Lemma 2.3. Another application of Proposition 2.2 completes the proof.

(3)⇒(1): Suppose there exists (r,s) in $\mathbb{Q}\times\Lambda(\widetilde{\alpha})$ such that $rst = 2\pi$. Then $\frac{1}{t} = \frac{r}{2\pi}s$ and so, by Proposition 2.2, A is not α_t-simple. □

As an example, we specialize to the case when A is prime to get the following.

Corollary 2.5. *Let $(A,\mathbb{R},\alpha)$ be a unital C^*-dynamical system. Assume that A is prime but not simple. If α is minimal then $A \rtimes_{\alpha_t} \mathbb{Z}$ is simple for all $0 \neq t \in \mathbb{R}$.*

Proof. Follows from Theorem 2.4 since $Z(A) = \mathbb{C}1$ and so $\Lambda(\widetilde{\alpha}) = \{0\}$. □

Theorem 2.4 (and Corollary 2.5) fails for simple C*-algebras, as one can see from the (C*-version of) the example of A. Kishimoto described after Corollary 1.5. We conclude this section with some open problems.

Problem 3: Is Proposition 2.2 true if we erase the condition on A being either commutative or prime?

Problem 4: Suppose that $(A,\mathbb{R},\alpha)$ is a minimal C* dynamical system and assume that A is simple. Characterize the values of t for which the crossed product associated to the time t automorphism α_t is a simple C*-algebra.

References

[1] I.P. Cornfeld, S.V. Fomin and Ya.G. Sinai. *Ergodic Theory.* Grundl. Math. Wiss. Vol. 245. Springer-Verlag, Berlin-Heidelberg-New York, 1982.
[2] B. Itzá-Ortiz. *Eigenvalues, K-theory and minimal flows.* Preprint 2004, arXiv:math.OA/0410426.
[3] A. Kishimoto. *Simple crossed products of C*-algebras by locally compact abelian groups.* Yokohama Math. J. **28** (1980), 69–85.
[4] D. Olesen and G. Pedersen. *Applications of the Connes spectrum to C*-Dynamical Systems.* J. Funct. Anal. **30** (1978), 179–197.
[5] D. Olesen and G.K. Pedersen. *Applications of the Connes spectrum to C*-Dynamical Systems II.* J. Funct. Anal. **36** (1980), 18–32.
[6] D. Olesen and G.K. Pedersen. *Applications of the Connes spectrum to C*-Dynamical Systems III.* J. Funct. Anal. **45** (1982), 357–390.
[7] G. K. Pedersen. *C*-algebras and their automorphism groups.* LMS Monographs 14, Academic Press, London/New York, 1979.
[8] E. Størmer. *Spectra of ergodic transformations.* J. Funct. Anal. **15** (1974), 202–215.

[9] S. Strătilă. *Modular theory in operator algebras*. Editura Academiei and Abacus Press, Bucureşti/Kent, 1981.
[10] M. Takesaki. *Duality for crossed products and the structure of von Neumann algebras of type* III. Acta Math. **131** (1973), 249–310.
[11] M. Takesaki. *Theory of operator algebras II*. Encyclopedia of Mathematical Sciences 125. Springer-Verlag, Heidelberg/New York, 2002.

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