Far East Journal of Mathematical Education
Volume 4, Number 2, 2010, Pages 93-115
This paper is available online at http://pphmj.com/journals/fjme.htm

MATHEMATICAL THINKING, CONCEPTUAL FRAMEWORKS: A REVIEW OF STRUCTURES FOR ANALYZING PROBLEM-SOLVING PROTOCOLS

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Abstract

Recently, it has been considered important to reflect on the coincidences between the mathematical thinking of the "School of Mathematics" and the "Discipline of Mathematics". It is widely accepted that a professional mathematician has naturally developed reasoning abilities that are essential to his practice. Such reasoning abilities are considered central to student

2010 Mathematics Subject Classification: 97E50, 97C30, 97C40.

Keywords and phrases: mathematical thinking, mathematical creativity, problem solving.

Communicated by Isabel Marrero

Received January 18, 2010; Revised May 8, 2010

learning at different educational levels. While the problem is not entirely new, the enrollment diminished in science and engineering university programs, suggests new reflections that might help the community when proposing scientific and engineering training programs. In this paper, we conduct an analysis of the processes developed by a mathematician during problem solving activities, with the aim of identifying relevant characteristics of the structure representations of mathematical concepts that he shows to solve problems. In the same lines, we pose the question: how this structure influences the mathematical creativity? [14]. The mathematics teacher, by knowing the characteristics of mathematical reasoning and the limitations to promoting them in the classroom, is better positioned to help his students in their learning processes.

Introduction

This study arose from an author's discussion around Dudeney's problem on the bisection of an equilateral triangle into four pieces to produce a square with the same area [10], see Figure 1. The problem led the authors to reflect on the discoveries of mathematical concepts by the Greeks, on mathematical thinking in general and on the problems in the mathematics classroom when promoting characteristics of mathematical reasoning. During the discussion, arose the need to specify those characteristics of mathematical reasoning which we implicitly agree on, but there had not been made explicit, and above all, on the possibility of promoting a change in the students' reasoning processes in the mathematics classroom.

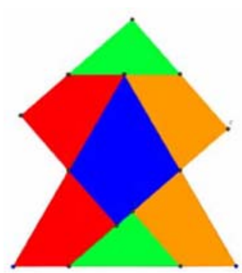

Figure 1. Dudeney's bisection of an equilateral triangle.

At this point, we decided to search in the literature sources to find out how do researchers adopt the views in order to better understand the different forms of reasoning and the different abilities that humans can develop. Mathematics educators are revisiting this subject (see [5]) because some researchers seem not to pay attention to the sources of the mathematical activity. Mathematicians are characterized by having developed forms of reasoning that have allowed scientific advances in general, and particularly in mathematics. According to Watson ([38, p. 4]), the reasoning processes involved in mathematical thinking are characterized by:

> Empirical exploration, logical deduction, seeking variance and invariance, selecting or devising representations, exemplification, observing extreme cases, conjecturing, seeking relationships, verification, reification, formalisation, locating isomorphisms, reflecting on answers as raw material for further conjectures, comparing argumentations for accuracy, validity, insight, efficiency and power. It is also about reworking to find errors in technical accuracy, and errors in argument, and looping actively for counterexamples and refutations.

The discussion promoted by Anne Watson (ibid) led us to reflect, from a global perspective, on the characteristics of mathematical reasoning and on its connection to what happens in the mathematics classroom. So as to gain a deeper insight into this problem, we decided to analyze a mathematician's ways of reasoning when confronted by the same types of activities that have been used by some researchers in mathematics education at different instruction levels.

While it is possible for some of the characteristics mentioned by Watson to be developed in the classroom, the social atmosphere in which each of them is implemented is completely different from that involved when developing the discipline. Along these lines, it is important to note Hadamard's classic book [14] and its discussion of how Poincaré describes the creative processes in mathematics, which could hardly be carried out in a mathematics classroom. By this we mean, that Hadamard and Poincaré assign great importance to the *incubation process*, (letting a problem "stay on the head", thus giving the subconscious a chance to work on possible ways to solve it) that is characteristic of mathematicians when solving problems. In the mathematics classroom, the student is restricted to working on a given problem, continuing on to another one and advancing along a program established by the educational institution. This limits the possibility of reflecting on a single problem for an extended period of time.

The stance adopted by Zazkis [35], in reaction to Watson's article, is that by recognizing the differences between the conditions under which mathematical reasoning takes place and the limitations to promoting it in the mathematics classroom, the mathematics instructor plays a key role to widening the intersection between a mathematician's and a student's way of reasoning. On this we agree with Zazkis.

In addition to the aforementioned works, we consulted various sources [14, 33, 24] so as to more accurately identify those aspects that might shed some insight into understanding mathematical thinking processes, including those related to creativity.

What is mathematical creativity and what role does it play in the learning processes?

In reading several references, we noted important aspects involving mathematical learning and its connection to creativity in mathematics, which led us to pose the following questions:

What does creative mathematical learning involve?

What characteristics of mathematical reasoning can be identified so that, appropriate learning tasks can guide its development process in a mathematics classroom?

Conceptual Elements

In order to analyze possible answers to the posed questions, it is relevant to review what some authors, including outstanding mathematicians [14, 24] have identified as characteristics of mathematical creativity. Sriraman ([33, p. 20]) proposes a general definition of what creativity is to him:

> I think it is sufficient to define creativity as the ability to produce novel or original work, which is compatible with my personal definition of mathematical creativity as the process that results in unusual and insightful solutions to a given problem, irrespective of the level of complexity.

This interpretation proves too general to an understanding of the processes involved in mathematical creativity.

For Hadamard ([14, Preface, p. x), mathematical creativity has four stages:

1. Preparation. You work hard on a problem, giving your conscious attention to it.

2. Incubation: Your conscious preparation sets going an unconscious mechanism that searches for the solution.

3. Illumination: An idea that satisfies your unconscious criteria suddenly emerges into your consciousness.

4. Verification: You carry out further conscious work in order to verify your illumination, to formulate it more precisely, and perhaps to follow up on its consequences.

This conception is directly related to that expressed by Poincaré [24].

The works of Polya [25] Mason [21] and Schoenfeld [29], among others on aspects of problem solving, do not include incubation or illumination, probably because these two characteristics are linked to the solution of a problem over a long period of time, which is not often the case in the mathematics classroom.

Sriraman [33] considering aspects like those pointed out by Hadamard, interviewed five mathematicians so as to analyze whether Hadamard's proposal was still relevant. He adopted the questionnaire in *L'Enseignement Mathématique* (quoted in Sriraman [33, p. 19]), expanding it with some new questions, and confirmed that the above conditions are at the core of what creativity is in mathematics. He also highlights how intuition, social interaction, the use of heuristics and the necessity of proof should be incorporated into the process of mathematical creativity. That is:

> This study has attempted to add some detail to the preparation-incubation-illumination-verification model of the Gestalt by taking into account the role of imagery, the role of intuition, the role of social interaction, the role of heuristic, and the necessity of proof in the creative process (ibid, p. 31).

It is important to note that the elements suggested by Sriraman [33] point out to the possibility of incorporating the ideas related to mathematical creativity and its learning. Along this same line of thought, Hadamard ([14, p. 104]) states:

> Between the work of the students who try to solve a problem in geometry or algebra and a work of invention, one can say that there is only a difference or degree, a difference of level, both works being of a similar nature (ibid, p. 104).

With this as their starting point, researchers in Mathematics Education and instructors face various challenges to incorporate the idea of creativity in mathematics to school mathematics. In this sense, it is of great importance to seriously consider Sriraman's proposal:

> I suggest that contemporary models from creativity research can be adapted for studying samples of creativity such as are produced by high school students. Such studies would revel more about creativity in the classroom to the

> mathematics education research community. Educators could consider how often mathematical creativity is manifested in the school classroom and how teachers might identify creative work. One plausible way to approach these concerns is to reconstruct and evaluate student work as a unique evolving system of creativity ([33, p. 32]).

With this perspective in mind, we think it is appropriate to consider questions such as:

Is it possible to justify that mathematical creativity has several levels from which students, in the classroom, only explore the basic ones in contrast to professional mathematicians?

How can these levels be identified in order to design learning activities that can stimulate mathematical creativity in students?

More specifically, what features should the learning activities possess so that students can manifest some of the aspects involved in the mathematical creativity process when solving problems?

Methodology

The aim of the experiment was to identify important elements in the mathematical reasoning processes developed by a mathematician, focusing the attention in the solution of school type problems. This was motivated by the discussion on Watson's paper ([38, p. 4]), since she refers to those characteristics to the mathematical thinking when developing the discipline and we consider the importance to record the processes when a mathematicians solve school type problems, since from this there are possibilities to reply his thinking in the classroom.

The experimental part of the work consisted in asking to a mathematician to solve a set of school type activities which would allow us to identify specific characteristics of his mathematical thinking. Our main objective was to find specific elements which could contribute to the design of learning tasks, whose solution processes, could help students to obtain learning with a wider mathematical thinking.

A set of 16 school type activities (problems or problematic situation) were proposed to the mathematician, which were chosen from several sources and whose characteristics have been recognized by some researchers to develop several types of abilities in the mathematical classroom and/or to promote reflection in pre-service teachers. Some of those activities had been considered as non-routine. Among those problems, solutions not always exist (for instance Q10, Appendix); in other instances, the statement was unusual to be considered as a problem, hence we wanted to know how was the mathematician's response to such a statement (Q12, ibid). In some others, a proof was required (Q15 and Q16, ibid) or the construction of a function with especial type of properties (Q4, Q8, Q9).

The experiment was carry out in three one hour sessions and the mathematician was allowed to only use paper and pencil; he was asked to speak out loudly while solving the problems, since there was a type recorder on his desk, which would provide valuable information to be analyzed later.

The analysis of the experiment was conducted under Schoenfeld's problem solving protocol ([29, pp. 292-297]) which considers four fundamental elements:

(1) Resources: Characterized by the propositional and procedural knowledge of mathematics.

(2) Heuristics: Constituted by a series of strategies and techniques for problem solving working.

(3) Control: Decision about when and what resources and strategies to use.

(4) Beliefs: Determined by the mathematical "world view" that determines how someone approach a problem.

Using this protocol, we were able to establish the elements that were search for in the experimentation process.

Analysis of Results

The mathematician provided us with a great deal of information on how he solved the problems, which he did correctly in every case. For the purposes of this paper, we will only show those reasoning processes used by him to solve the activities that we consider to be of interest.

First, the mathematician was asked to solve the following exercise in his mind (this exercise was taken from Glaeser's anecdote [7, p. 43]).

Question 1

Mentally calculate $\dfrac{10^2+11^2+12^2+13^2+14^2}{365}$.

The mathematician first found the following relationship: $10^2+11^2+12^2 = 365 = 13^2+14^2$, and gave an answer of "two" to the activity. However, he asked himself the following question verbally: "Given five consecutive numbers, when is the sum of the squares of the first three numbers equal to the sum of the squares of the last two?" This is an important point that involves the possibility of questioning about what one is doing and the feasibility of asking oneself if it is possible to ponder a process in order to devise *a new way of thinking*, resulting from the activity itself and arising from the process of solving the initial problem. This form of mathematical reasoning can be identified as what Perkins and Simmons [23] consider the inquisitive structure (see below for a discussion of their structure) which, in this case, the mathematician has constructed through his daily activity.

This reformulation process led the mathematician to consider this new problem, leaving behind, so to speak, the specific case: Let n be a natural number, is there an integral solution to the equation $n^2+(n+1)^2+(n+2)^2 = (n+3)^2+(n+4)^2$?

Before analyzing how the mathematician approached this new problem, we show the process followed to answer the next question, where he was specifically asked to generalize the result. We will then analyze the processes devised by the mathematician for both activities, since they bear a certain similarity.

Question 3

(a) How would you calculate the sum of 10 consecutive numbers? For example, as in the case of $10+11+12+13+14+15+16+17+18+19 = 145$.

(b) How can this result be generalized and how could you justify your conjecture?

The mathematician solved this problem by adding each term, without a predefined strategy. A different strategy would be, for instance, writing

$$(10+0)+(10+1)+(10+2)+\cdots+(10+9) = 10*10+(0+1+2+\cdots+9),$$

which he used to solve part (b) but not part (a). In other words, for part (b) he used $n+(n+1)+(n+2)+\cdots+(n+9)$, obtaining $5(2n+9)$, which he then calculated for $n=10$, whose result he verified to be that obtained in (a). He then tackled the question of how to find the sum of k consecutive whole numbers. He immediately formulated the problem algebraically, writing out $n + (n + 1) + (n + 2) + \cdots + (n + (k - 1))$, which he re-wrote as $\frac{k(2n + k - 1)}{2}$. He then calculated the result for $n = 10$ and $k = 10$ in his head to check his result with that for (a). He asked if we wanted a proof by mathematical induction for the equation in (b), to which we replied that, for the moment, it was not necessary.

This allows us to extract some characteristics of the mathematician's thought processes when solving the two problems:

- Structured solution of the first problem, classical arithmetic sum algorithm for the second.
- The reasoning used by the mathematician in the first problem afforded him the chance to verbally state one possible generalization; in the second, he did not need to consider the generalization since this was included in the question (part b).
- Conversion of the first problem statement from a verbal to a verbal-algebraic form.
- Solution of the equation posed in the first question (single solution for $n = 10$). Solution of the second problem through an algebraic process.

The work done by the mathematician with these two questions reveals some of the characteristics expressed by Watson [38] on the mathematical endeavor. Trying to understand these ways of thinking led us to reflect on the cognitive model of Perkins and Simmons [23], in which they propose four cognitive structures important for analyzing teaching and learning processes in mathematics, science and programming. The structures they propose are:

- Content frame.
- Epistemic frame.
- Problem-Solving frame.
- Inquiry frame.

The content frame has received the most attention since long ago, and it is obvious why, since a certain basic knowledge is necessary in order to develop more complex abilities. In the case of mentally calculating the sum $\frac{10^2 + 11^2 + 12^2 + 13^2 + 14^2}{365}$, the content frame is tied to squaring a whole number, adding whole numbers and division of whole numbers. The epistemic frame is what controls the development of the activity (the coherence of the action). The problem-solving frame allowed the mathematician to associate the sum of the squares of the first three consecutive numbers and compare it to the sum of the squares of the other two consecutive numbers. This strategy, along with the inquiry frame, allowed him to formulate a conjecture regarding the possibility of generalizing the problem.

Perkins and Simmons [23] note that it is precisely the epistemic and inquiry frames which have received the least attention in the development of mathematical skills in the classroom. Hiebert et al. [15], in reviewing the mathematics curriculum in the USA, consider that researchers have managed to permeate the curriculum with features related to problem solving, but not to the development of other types of skills, such as those mentioned above.

In analyzing the mathematician's work from a representational point of view [11, 12], we see the mathematician's evident skills with a well structure registers representation. In this case, in the first question, using a mental calculation he processes in an arithmetic register, then shifts from an arithmetic register to a verbal one and from there to an algebraic one. The processing in the algebraic register is what enables him to obtain an answer to his conjecture. According to Duval's theoretical frame, the articulation between representations is fundamental to the mathematical endeavor. In the second question, one aspect of the control of his actions stands out. When solving part (b), after obtaining the algebraic expression for calculating the sum of k consecutive numbers, he solved the specific case for $n = 10$, which he compares to the result for (a). This allows him to feel that the "process being developed is on the right track".

Question 2

Calculate the area of a square inscribed within a unit circle.

For this question, the mathematician drew a picture of a square inscribed within a circle. His drawing made him consider the unknown "x". It was probably through a visualization process that he was able to picture a right triangle with a hypotenuse equal to 1. Undoubtedly recalling the Pythagorean Theorem, he called the base "y", without immediately searching for the relationship with the unknown "x" (see

Figure 2). He wrote the algebraic expression relating the application of the Pythagorean Theorem, and right there, he looked for the missing relationship between "x" and "y". He found the value for the side of the square and calculated the area requested. At the conclusion of this process he stated only that he performed a mental calculation for the area of the circle $(\text{area} = \pi)$ to compare it with the area of the rectangle (2) to ensure his process had resulted in a coherent answer.

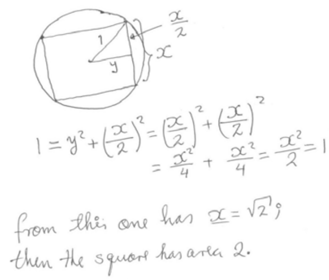

Figure 2. The role of visualization to solve the problem.

Analysis of the Process Followed in Solving the Problem

From an overall standpoint, we can say that the mathematician used the drawing to represent the situation and to enable him to add notes about the variables he was going to use. It is interesting to note how his figure is an intermediate step before immediately proceeding to an algebraic treatment. Moreover, we can say that the articulation between representations was fundamental. In this case, there was a figural representation and conversion and treatment process, in keeping with Duval's theoretical aspects. Specifically, we can mention certain characteristics in his solution process:

- Conversion from verbal statement to a figural representation to portray the situation.
- Mathematical visualization process to draw the radius of the circle as the hypotenuse of a right triangle.
- Use of letters to depict the unknowns in the figural representation of the situation.
- Conversion of figural representation to an algebraic representation.
- Algebraic treatment to find the answer requested.
- Verification process.

In keeping with Sriraman's observations [33], mathematical visualization plays a key role in problem solving. In the case at hand, this element is fundamental before proceeding to an algebraic treatment. In mathematics education, visualization has been a constant theme of the research agenda (see, for example, [4, 34, 38]) and many researchers consider it as an important skill in both, the learning and advancing of mathematics.

Going back to the problem at hand, we see that the mathematician uses the figural representations as an intermediate step between the verbal and algebraic representations. He used this skill set and only relies on the mathematical visualization process depending on the situation, but it is obvious that what interests him the most, is advancing immediately to the algebraic process.

An important element to consider is the mental calculation done by the mathematician to compare the areas of the circle and square. From the point of view of the Perkins and Simmons theoretical aspects, the mathematician's epistemic frame allows him to follow the actions of the algebraic treatment, which in the event that a contradictory result is reached (formal contradiction), gives the mathematician a control element (the comparison of the two areas) that allows him to realize the contradiction (cognitive contradiction) and which would serve as a trigger to analyze the algebraic process to detect the mistake (not applicable in this case). Observing these characteristics in the solution processes displayed by the mathematician gives rise to the question:

How to develop, in students, these aspects to control their actions as well as sensitivity to contradiction when discussing tasks? Saboya's Ph.D. thesis (under review) attempts to shed light on this problem in secondary students. The design of learning activities by the instructor, taking into account this critical point and his intervention in the classroom, are essential, as Saboya [28] argues.

In solving the other problems, the mathematician rarely relied on figural representations, this being an evidence that he has constructed a structure of the different representations of mathematical concepts, that allows him to quickly assess the one most suited to the problem. In the cases we observed, that the mathematician proceeded as quickly as possible to the algebraic processes.

This point merits further consideration. When forming concepts and solving problems, we, like Zimmermann and Cunningham [37], Eisenberg and Dreyfus [4], Vinner [34] and Sriraman [33] believed that the visualization processes are of the utmost importance. For example, Zimmermann [36] maintained that students who have not developed this skill would be incapable of succeeding in a calculus course requiring that non-routine problems be solved. The work by Selden et al. [30-32] substantiates this assertion by way of research that shows how average students (neither brilliant nor poor), who had passed a calculus course, were not able to solve a questionnaire with non-routine problems in which visualization could add elements to aid in their solution.

As we mentioned before, the mathematician solved the problems by relying on a minimum number of figural representations. It is, however, interesting to analyze one solution to one of these problems where he proceeded very differently (see results with teachers in Hitt [16]).

Question 8

Construct two different real-valued functions such that $f : \mathfrak{R} \rightarrow \mathfrak{R}$, $g : \mathfrak{R} \rightarrow \mathfrak{R}$ which satisfy:

$$f(1 + x) = f(1 - x), \quad \forall x \in R; \quad g(1 + x) = g(1 - x), \quad \forall x \in R.$$

The immediate answer provided by the mathematician after reading the statement was that $f(x) = 2, \ \forall x \in R$ and $g(x) = 3, \ \forall x \in R$. Since he found those two functions rather quickly, we asked him to provide two functions of a different nature, that he could consider one of them as $g(x) = c$ (constant) and to supply another one of a different nature. He stated how that “was less obvious.”

What a interesting is that he drew a graph and reflected on it for a few minutes, after which he drew a second graph and provided the answer in algebraic notation $f(x) = |\,x - 1\,|$ (see Figure 3).

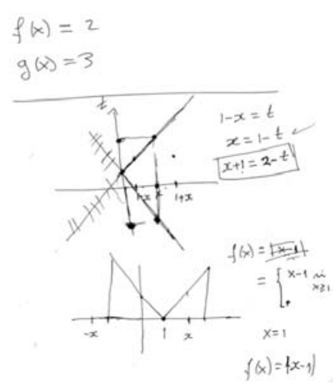

Figure 3. Algebraic and functional representation in the solution process.

When we inquired about his reasoning, since we did not understand the first figural representation, he stated that the first representation was not a function that he merely wanted to see the property expressed in the statement. We were surprised at how his "reading of his representation was with the page sideways" (in fact, when he explained his reasoning, he rotated the paper 90 degrees counterclockwise). Since the first variable he wrote in his figure was "x", writing $1-x$, x, $1+x$ along the horizontal axis and the variable t along the "y", he added that upon writing $1-x=t$, the transformation caused him to see the vertical axis as the horizontal one and vice versa, and that once he rotated the axes, he was only interested in the top part. On turning the page, the direction of the "t-axis" is toward the left, and when he draws his second representation, the mathematician is representing the function he found with the axes in their standard orientation.

This moment is crucial to our analysis of the mathematician's problem-solving approach. In this situation the mathematician, when asked about other types of functions, followed an unusual path to find the functions requested. In summary, we can propose the following characteristics of his solution process:

- Reading the statement and immediately producing an answer with two constant functions.
- Producing an algebraic representation followed by a functional one in response to a request by one of the researchers.
- Mathematical visualization process using his functional representation.
- Drawing of a graphical representation.
- Conversion to an algebraic representation.

At the request of one of the researchers to provide two functions of a different type, and given his inability to directly construct the functions requested, he created an algebraic representation followed by a "spontaneous representation" (functional representation) that allowed him to progress toward the problem solution. The spontaneous representation he showed when answering the question is tied to the action (an *habitus* as expressed by Bourdieu [3]), and is a functional representation in the sense that it allows one to progress toward the problem solution (see [19, 20]).

Functional representations are spontaneous representations that are used when solving problems. Depending on the statement, the representations can be either institutional (those used by teachers and found in books, shown on a screen using software, etc.) or they can be functional (see [18-20]), which are characterized as spontaneous, non-institutional, representations and which allow one to continue with the problem-solving process at hand (Duval [13] calls them *transitional representations*). In the case that concerns us, the mathematician is using an unusual representation, and yet one that allows him to finish solving the problem.

With the remaining problems, the mathematician scarcely resorted to figural representations. He tended to proceed as quickly as possible to algebraic representations for processing.

Discussion

An initial approach to the problem presented was provided by Hadamard's [14] work which, along with the analyses made by Sriraman [33] and Watson [38] provided us with some of the characteristics detected in the problem-solving processes employed by mathematicians. This serves as a general response to the question we posed. The structures mentioned by Perkins and Simmons [23] are less global and we considered them as well to demonstrate explicit problem-solving characteristics. In our case, we wanted to explain the heuristic processes that include mathematical visualization, control of actions taken, the importance of functional representations in problem solving and conversion processes among representations. From this point of view, a relevant aspect is the articulation between representations as an essential element in solving non-routine problems. Also, while this approach is indeed important, not many studies are available on the articulation between representations when solving non-routine problems.

The analyses of Sriraman [33] and Watson [38] on the characteristics of the mathematician's thinking and that of the students, demonstrate the differences that have to be taken into account. As we noted earlier, we agree with Zazkis's [35] concerning the importance to the mathematics teacher that he should be familiar with some of the reasoning characteristics used by mathematicians, which could help him to promote advanced skills in his students when presenting work in the classroom. The examples discussed show that the mathematician very quickly progresses to algebraic representations, to which he assigns a greater priority. This could be due to the fact that he has undergone a training process that has allowed him to construct a structure through the use of different representations, allowing him to quickly select the most suitable one. This is an important point to emphasize, since the construction of mathematical concepts in students requires the systematic use of the different representations so that the advantages offered by each can be used to their fullest (see [8]). This aspect should be considered a priority in the teaching of mathematics since it is stressed by only a few researchers.

Another point to consider is the confidence with which the mathematician carries out his proofs and the construction of counterexamples. When training students, it is important to promote teamwork in problem solving, to develop the proof as expressed by Balacheff [1] and, through group discussions, to establish the importance of the proof and use of counterexamples (as promoted, for example, by the SiRC group in research situations in the mathematics classroom, see Grenier and Payan [9]).

Closing Remarks

It is important to note that the experiment presented in this paper, using activities that were employed in other studies, shows that the cognitive processes of a mathematician, when solving problems, reveal the existence of certain reasoning patterns (construction of examples and counterexamples, generalization, use of control strategies). We mention these characteristics in the answer to the question that led to the reflection on the mathematician's routine activities and that of the student in the mathematics classroom. We found different characteristics of a mathematician's thinking that are related to the visualization processes, as mentioned in previous paragraphs. In general, we have seen how the mathematician rapidly progresses to the use of algebraic representations. From the standpoint of mathematical learning, the different representations and the way they are structured play an essential role in the understanding and construction of mathematical concepts (Duval [12]). Hence it is necessary to promote the use of different representations of mathematical objects when solving problem situations or problems.

Our approach to creativity, conceived in the mathematics classroom, *is a student's ability to confront a non-routine mathematical activity* (*problem situations or open problem*) *and which allows him to mathematically visualize the problem so as to find ways to solve or advance in the process of solving the question asked*. These processes can involve complementary reasoning linked to an inquisitive internal process on the possible generalization of the results. The referred mathematical visualization process is in agreement with the position of Zimmermann and Cunningham [37] and is connected to the production of representations in the mind, on paper, or on a calculator or computer screen, that helps to solve the activity proposed to or by the student.

Concerning the role of creativity in the learning process and according to our definition, mathematical creativity is linked to solving a problem situation or problem (possibly of the open variety) that gives rise in the students, to mathematical visualization processes. These processes are tied to what we have called divergent thinking, characterized by, among other things, its richness of ideas, its depth in terms of searching for relationships and its role in unifying representations and concepts. One characteristic of divergent thinking is the generalization process, which involves an aspect of what Perkins and Simmons [23] have referred to as the inquiry frame. Although it is present at a basic level in the mathematics classroom, it does not seem to be practiced systematically by teachers in the student learning process. It is, however, possible to develop its evolution by presenting them with non-routine activities that are within their reach. Advances along these lines have been made, for example, by the Grenoble SiRC group, which is searching for activities that promote the construction and evolution of this type of cognitive structure or that of Barallobres [2].

We argue that creativity can be developed in students. In this process, however, one of the greatest problems is that students, while in the early years of compulsory education, may not have been presented with an opportunity to develop their creative potentials. The increased focus on solving open-type problems and the so-called problem situations has resulted in a shift toward the plausible development of creativity in the mathematics classroom. An important element, from the classroom work perspective, is that of creating a balance between the use of different activity types that lead to the construction of the different cognitive structures discussed over the course of this paper. In other words, a balance in the use of problem situations, open problems, closed problems, exercises and, finally, the process of institutionalizing knowledge might help students constructing knowledge from a creativity point of view. Related with this point of view, the closed problem and the exercise should be used to consolidate knowledge and, thus, to promote convergent thinking.

Another relevant aspect related to mathematical creativity is that the mathematician, when confronting a complex problem, may be required to engage in incubation processes over a long period of time, as described by Hadamard [14]. This, in principle, poses constraints in both, time and space to its applicability in the classroom. One possibility for the student to experiment with this could be to propose research projects associated with problems whose solution timeframe could be longer (e.g., Mounier and Aldon [22]). One of the teacher's tasks would be to promote regular discussions until the solution is found. In order to develop shorter term incubation processes in the classroom, the students could be asked to solve problematic situations or problems that, once solved in class, could be assigned to them as homework (individual work) so that they could find, first, new ways to solve the same activity and, second, possible ways to generalize the problematic situation or problem.

Acknowledgments

Our research was made possible thanks to funding from the Fonds de la Recherche sur la Société et la Culture du Québec (Project #008-SE-118696) and the Conseil de Recherches en Sciences Humaines du Canada (Project #081159). The second author acknowledges the support received from CONACyT through research project number 61996 and from the PIFI 3.5 funds, “Cuerpos Académicos”. Some funding was provided by the Spain’s Ministry of Science and Innovation’s National Research Plan, reference EDU2008-05254.

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Appendix

Question 1 ([7])

$$\text{Mentally calculate } \frac{10^2+11^2+12^2+13^2+14^2}{365}.$$

Question 2

Calculate the area of a square inscribed within a unit circle.

Question 3 ([2])

(c) How would you calculate the sum of 10 consecutive numbers? For example, as in the case of

$$10+11+12+13+14+15+16+17+18+19=145.$$

(d) How can this result be generalized and how would you solve your conjecture?

Question 4 ([17])

Construct three different functions f_1, f_2 and f_3, whose domain is R and which satisfy:

$$|f_1(x)|=|f_2(x)|=|f_3(x)|=2, \quad x \in R.$$

Question 5 ([26])

A boy walks from his home to school in 30 minutes. It takes his brother 40 minutes. His brother left the house five minutes before he did. In how many minutes will he reach his brother?

Question 6 ([26])

It takes a train 15 seconds to pass by a telephone pole from beginning to end, and it takes it 45 seconds to completely traverse a 540 meter long tunnel. What is the train's speed in meters per minute, and what is its length in meters?

Question 7 ([16])

Construct two different functions f, g, whose domain is R and which satisfy:

$$f(f(x))=1; \quad g(g(x))=1, \quad x \in R.$$

Question 8 ([16])

Construct two different real-valued functions such that $f : \mathfrak{R} \to \mathfrak{R},\ g : \mathfrak{R} \to \mathfrak{R}$ which satisfy:

$$f(1 + x) = f(1 - x), \quad x \in R; \qquad g(1 + x) = g(1 - x), \quad x \in R.$$

Pregunta 9 ([16])

Construct two different real-valued functions such that $f : \mathfrak{R} \to \mathfrak{R},\ g : \mathfrak{R} \to \mathfrak{R}$ which satisfy:

$$f(x + 1) = f(x - 1), \quad x \in R; \quad g(x + 1) = g(x - 1), \quad x \in R.$$

Question 10 ([27])

A coffee and three pastries cost 2.70 Euros at a restaurant. Two cups of coffee and two pastries cost 3 Euros. Three cups of coffee and one pastry cost 3.50 Euros. Find the price of a cup of coffee and one pastry.

Question 11 (Pluvinage, unpublished)

Given the function $f(x) = 3x,\ x \in R$ is there a quadratic function $g(x)$ such that the quadratic function is tangent to the straight line a $x = 1$?

Question 12 (Lupiañez, unpublished)

The following takes place in a mathematics classroom:

- Teacher: “How old are you Aureliano?”

- Aureliano: “13”

- Teacher: “Ha! If I add the ages of my three children, I get your age. Knowing that the product of their ages is 36, can you figure out my children’s ages?”

- Aureliano ponders this and replies: “No, I need one more piece of information.”

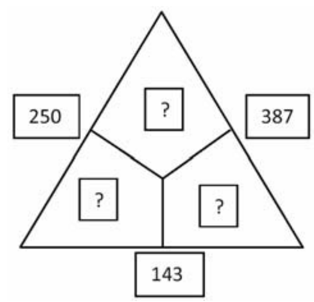
- The teacher says: “You are right, I forgot to mention that my son, the oldest, has black hair.”

- Aureliano: “Ha! Then I know the answer.”

- What did Aureliano do to determine the ages of the teacher’s three children?

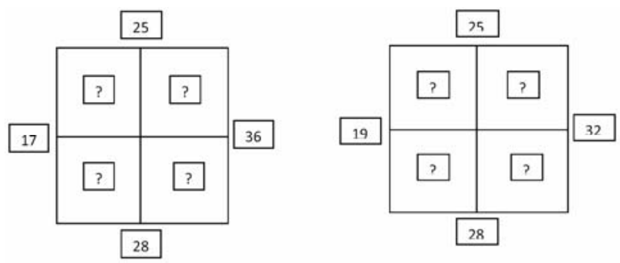
Arithmagons 13 ([39])

The sum of two numbers inside each side of the triangle is equal to the number outside. Find those numbers.



Arithmagons 14 ([39])

The sum of two numbers inside each side of the square is equal to the outside number. Find those numbers.



Question 15 ([17])

Let *f* and *g* be two functions of $\mathfrak{R}$ in $\mathfrak{R}$. Assuming that $(f(x))^2 + (g(x))^2 = 0$ for every $x \in \mathfrak{R}$, does that imply that $f(x) = 0$ and $g(x) = 0$ for every $x \in \mathfrak{R}$?

Question 16 ([17])

Let *f* and *g* be two functions of $\mathfrak{R}$ in $\mathfrak{R}$. Assuming that $(f(x))(g(x)) = 0$ for every $x \in \mathfrak{R}$, does that imply that $f(x) = 0$ or that $g(x) = 0$ for every $x \in \mathfrak{R}$?