

Formulating mathematical conjectures in learning activities, assisted with technology

Fernando Barrera Mora
barrera@uaeh.edu.mx

Aarón Reyes Rodríguez
aaron.reyes.rdz@gmail.com

Universidad Autónoma del Estado de Hidalgo

Summary

What types of activities should professional development programs include to revise and extend high school teachers' mathematical and pedagogical knowledge? We propose a route to engage high school teachers in an inquiry approach to reflect on their current practice and to construct hypothetical learning trajectories that can eventually guide or orient the development of their lessons. In this report, we focus on the activities that were worked within a professional community that include the participation of mathematicians, mathematics educators and doctoral students.

Introduction

What mathematical and pedagogical knowledge should the education of high school mathematics teachers include? Who should participate in the educational programs to prepare mathematics teachers? What should be the role of mathematics departments or the faculty of education in preparing prospective and practicing teachers? What types of educational programs should practicing teachers participate in order to revise and extend their mathematical knowledge and to incorporate research results from mathematics education into their practices? Traditional ways to prepare high school teachers normally involve the participation of both mathematics departments and the faculty of education. Mathematics departments offer courses in mathematics while the faculty of education provides the didactical or pedagogical courses. This model of preparing teachers has not rendered solid basis to help teachers provide an instructional environment in which they exhibit mathematical sophistication to interpret and prompt students' responses and to organize and implement meaningful learning activities for their students. Indeed, it is common to read that university instructors complain that their first year university students lack not only fundamental mathematical knowledge; but also strategies or resources to solve problems that require more than the use of rules or formulae.

Many practicing teachers, for different reasons, have not learned some of the content they are now required to teach, or they have not learned it in ways that enable them to teach what is now required. ...Teachers need support if the goal of mathematical proficiency for all is to be reached. The demands this makes on teacher educators and the enterprise of teacher education are substantial, and often under-appreciated (Adler, et al., 2005, p. 361).

Davis and Simmt (2006) suggest that teachers' preparation programs should focus more on teachers' construction of mathematical ideas or relations to appreciate their connections, interpretations, and the use of various types of arguments to validate and support those relations, rather than the study of formal mathematics courses. Thus, the context to build up their mathematical knowledge should be related to the needs associated with their instructional practices. "... [mathematical knowledge] needed for teaching is not a watered version of formal mathematics, but a serious and demanding area of mathematical work" (Davis and Simmt, 2006, p. 295). In this work, we report that teachers' mathematical knowledge can be revised and enhanced within an interacting intellectual community that fosters an inquisitive approach to

develop mathematical ideas and to promote problem-solving activities. The core of this community should include mathematicians, mathematics educators, and practicing teachers. This community promotes collaborative work to construct potential learning trajectories to guide or orient the teachers' instructional practices. Teachers need to be interacting within a community that supports and provides them with collegial input and the opportunity to share and discuss their ideas in order to enrich their mathematical knowledge and problem solving strategies. In this context, we illustrate the importance of using computational tools to represent and explore various ways of approaching mathematical tasks.

Research questions

Several research works (Santos-Trigo, 2004; Schoenfeld, 1994, 2000; NCTM, 2000) emphasize the importance of formulating and validating conjectures when learning and developing mathematics. Conjecturing processes involve several dimensions when technology is used systematically. For instance, the idea of generalizing is widely amplified and at the same time one has the opportunity to ask questions about the way in which a particular computational system works. The questions that guided this research are: What type of mathematical reasoning might be developed by high school teachers in order to reconstruct or enhance their mathematical knowledge when using technology to explore hypothetical learning trajectories? What type of mathematical arguments might high school teachers use to explain unexpected computer mathematical results?

Conceptual Framework

The conceptual framework is structured around two main theoretical issues: (i) problem solving and technology and (ii) hypothetical learning trajectories. We have chosen these constructs since learning mathematics is achieved through problem solving, which is enhanced by using technological tools. In this regard, we argue that promoting an inquiring approach when learning mathematics can be attained effectively by formulating questions and elaborating conjectures systematically. This path is strongly related with the finding and exploration of different hypothetical learning trajectories.

Problem solving and the use of technology

In problem solving activities it has been recognized the relevance of an inquiring or inquisitive approach for teachers to work on mathematical tasks or to think of and reflect on their instructional activities. In this context, learning mathematics or developing mathematical knowledge through problem solving is conceptualized as working with tasks where:

A task, or goal-directed activity, becomes a problem (or problematic) when the “problem solver” (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation (Lesh and Zawojewski , 2007, p. 782).

It is important to clarify what is understood by a productive way of thinking. According with the same authors “...Developing a ‘productive way of thinking’ means that the problem solver needs to engage in a process of interpreting the situations, which in mathematics means modeling” (p. 782).

In this context, an important component is to develop an inquiring way of thinking to formulate questions, to identify and investigate dilemmas, to search for evidences or information, to discuss

solutions, and to present or communicate results. This means willingness to wonder, to pose and examine questions, and to develop mathematical understanding within a community that values both collaboration and constant reflection. At this point Schoenfeld (1994) argues: “Mathematicians develop much of that deep mathematical understanding by virtue of apprenticeship in to that community [mathematical community]—typically in graduate school and as young professionals” (p. 68). A mode of inquiry involves necessarily the challenges of the status quo and a continuous re-conceptualization of what is learned and how knowledge is constructed.

[In a community of inquiry] participants grow into and contribute to continual reconstitution of the community through critical reflection; inquiry is developed as one of the forms of practice within the community and individual identity develops through reflective inquiry (Jaworski, 2006, p. 202).

Taking this view into account, and considering that the use of technology has been playing an important role in the process of mathematical learning by enhancing different elements of mathematical thinking, particularly formulating and validating conjectures, it is relevant to ask: what is the role of a computer system in the process of posing and justifying conjectures? How trustable are the results obtained with the aid of a computer system?

Concerning the first question Santos (2007) argues: “A relevant aspect when representing a task with the aid of a dynamical software is that students have the opportunity to pose questions about the structure of some elements of the configuration” (p. 124).

Regarding the second question, Dick (2007) has introduced the term *Mathematical Fidelity* “to emphasize that the mathematics of the tool does not always represent the mathematics as it is understood by the mathematics community” (p.1174). In the example that we will discuss, it will be pointed out the strong necessity of providing mathematical arguments to deal with discrepancies between the computers results and the expected ones.

Hypothetical Learning Trajectories

To promote the teachers’ inquiring approach to their practice we rely on the construction of Hypothetical Learning Trajectories (HLT). These trajectories emerge from examining potential routes of solution of mathematical tasks. Simon and Tzur (2004) state that the construction of hypothetical learning trajectories is based on the following assumptions:

1. Generation of an HLT is based on understanding of the current knowledge of the students involved.
2. An HLT is a vehicle for planning learning of particular mathematical concepts.
3. Mathematical tasks provide tools for promoting learning of particular mathematical concepts and are, therefore, a key part of the instructional process.
4. Because of the hypothetical and inherently uncertain nature of this process, the teacher is regularly involved in modifying every aspect of the HLT (p. 93).

In this perspective, we suggest that teachers together with other community members (including mathematicians), work on various ways to approach the tasks and to identify relevant concepts and problem solving strategies needed to solve them. We argue that this type of teachers’ interaction becomes relevant to foster and to develop a problem solving approach that involves:

Seeing the mathematical content in mathematically unsophisticated questions, seeing underlying similarity of structure in apparently different problems, facility in drawing on different mathematical representations of a

problem, communicating mathematics meaningfully to diverse audiences, facility in selecting and using appropriate modes of analysis (“mental”, paper and pencil, or technological), and willingness to keep learning new material and techniques (Cohen, 2001, p. 896).

In addition, we also recognize that the use of computational tools offers to teachers the opportunity to enhance relevant aspects of mathematical thinking as well as to represent and examine mathematical tasks in terms of questions that can lead them to develop or reconstruct some mathematical results. For instance, the use of a dynamic software allows teachers to represent problems dynamically in order to recognize and explore mathematical relations within a geometrical configuration, and to identify loci described by members of the configuration when others are moved. In this context, the use of computational tools becomes important for teachers to discuss pedagogical paths associated with the hypothetical learning trajectories that can be useful to guide or orient their instructional practices.

We claim that the inquiring process is strongly intertwined with the appearance of hypothetical learning trajectories derived from a problem solving activity. By this we mean that in the process of formulating questions, there arises the opportunity to learn or reconstruct new mathematical concepts that emerge while pursuing those questions.

Research Design, Methods and General Procedures

The Center for Research and Applied Mathematics, that is part of a public university, is in charge of developing and implementing a professional program to revise and improve high school teachers’ mathematical and didactical knowledge. As a part of the program, we coordinated a 40 hrs instructional module, out of four, whose main aim was to illustrate and discuss the strengths and limitations of using computational tools in problem solving activities. To this end, a group that includes two mathematicians, one mathematics educator, and two doctoral students met together during two months, in sessions of three hours a week, to select and discuss the tasks that later would be used during the development or implementation of the activities with the teachers. During each session, one member of the group presented one problem and provided information related to its relevance or rationale and ideas about possible solutions. Then, all participants became engaged in the solution process and at the end they discussed and summarized the main ideas and ways used to approach the task. In this report, we focus on presenting general features of hypothetical learning trajectories that emerged as a result of working on those tasks that later we used to structure the development of the sessions with the actual group of teachers who participated in this module. Our unit of analysis is the work done and reported by the participants (the mathematicians, the math educator and the doctoral students) as a group. Thus, in presenting the results we use the word group to identify and characterize that work. The selected problems came from textbooks and research articles. There were also those that include the construction of an initial dynamic configuration (using dynamic software) in which teachers could formulate their own questions or problems.

The task: This problem involves an extension of a task discussed in Santos, et al. (2006, p. 125). In particular, the working group constructed a hypothetical route for teachers to develop an inquiring approach to the tasks in which the use of technology is encouraged. The problem arises from analyzing invariance and structure of simple components of a geometric configuration in order to identify an instructional path to foster the teachers’ construction of mathematical relations.

Given a straight line L , a point P in L and a point Q not in L , draw the segment PQ , a line L_1 perpendicular to PQ through Q and a line L_2 perpendicular to L through P . Call R the intersection of L_1 and L_2 . What is the locus of R when P runs on L ?

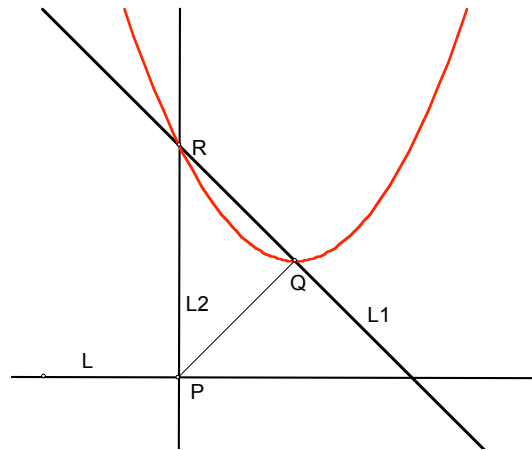


Figure 1: What is the locus of point R when point P is moved along line L ?

Research Results and Discussion

Figure 2 shows the different stages that appeared during the solution process of the task as well as main features that guide the construction of the different hypothetical learning trajectories. It should be mentioned that the diagram shows only one instructional route that the group proposed during the discussion of the task, however there were two more possible routes (finding the triangle with minimum area and the case where the locus is a hyperbola), whose discussion is not presented.

In this perspective, the meaning associated with the main stages that characterize the potential instructional trajectory involves: (i) the recognition of the high school teachers' knowledge base to represent and explore the initial task, (ii) the recognition that the aim of the developed task is to provide conditions in order that high school teachers reinforce and reconstruct their mathematical concepts in such way that this would help them to design and guide learning activities in the classroom, (iii) the discussed problem arises from analyzing minimal elements in a geometric configuration with the objective of designing learning tasks and (iv) the possibility that the teachers will bring into the discussion additional elements to modify every aspect of the hypothetical learning trajectory after they have solved the task.

One of the members of the discussion group suggested to approach the problem using Cabri-Geometry to construct the geometric configuration, after this, using the tool *Locus*, it was asked the software to describe the locus drawn by point R when P moves on L . Cabri-Geometry shows a graph that looks like a parabola, Figure 1. With this information, some of the members of the group went further in conjecturing, using Cabri-Geometry's tool *Equation or coordinate*: the equation of the locus described by R corresponds to a parabola. At this point there was consensus that formal arguments were needed in order to continue with the analysis to find connections and generalizations.

Using a coordinate system. An algebraic approach becomes important to construct an argument to show that the locus is a parabola. Here, the group used a Cartesian System in a proper position to facilitate algebraic operations.

Without loss of generality, one can assume that L coincides with the x axis, $P = (t, 0)$ and $Q = (a, b)$. In order to determine the coordinates of R , one finds the equations of L_1 , which turns

out to be $y - b = ((t - a)/b)(x - a)$, the equation of L_2 is $x = t$. Solving the system determined by these two equations yields $y - b = (x - a)^2/b \dots (*)$, which is in fact the equation of a parabola since a and b are fixed.

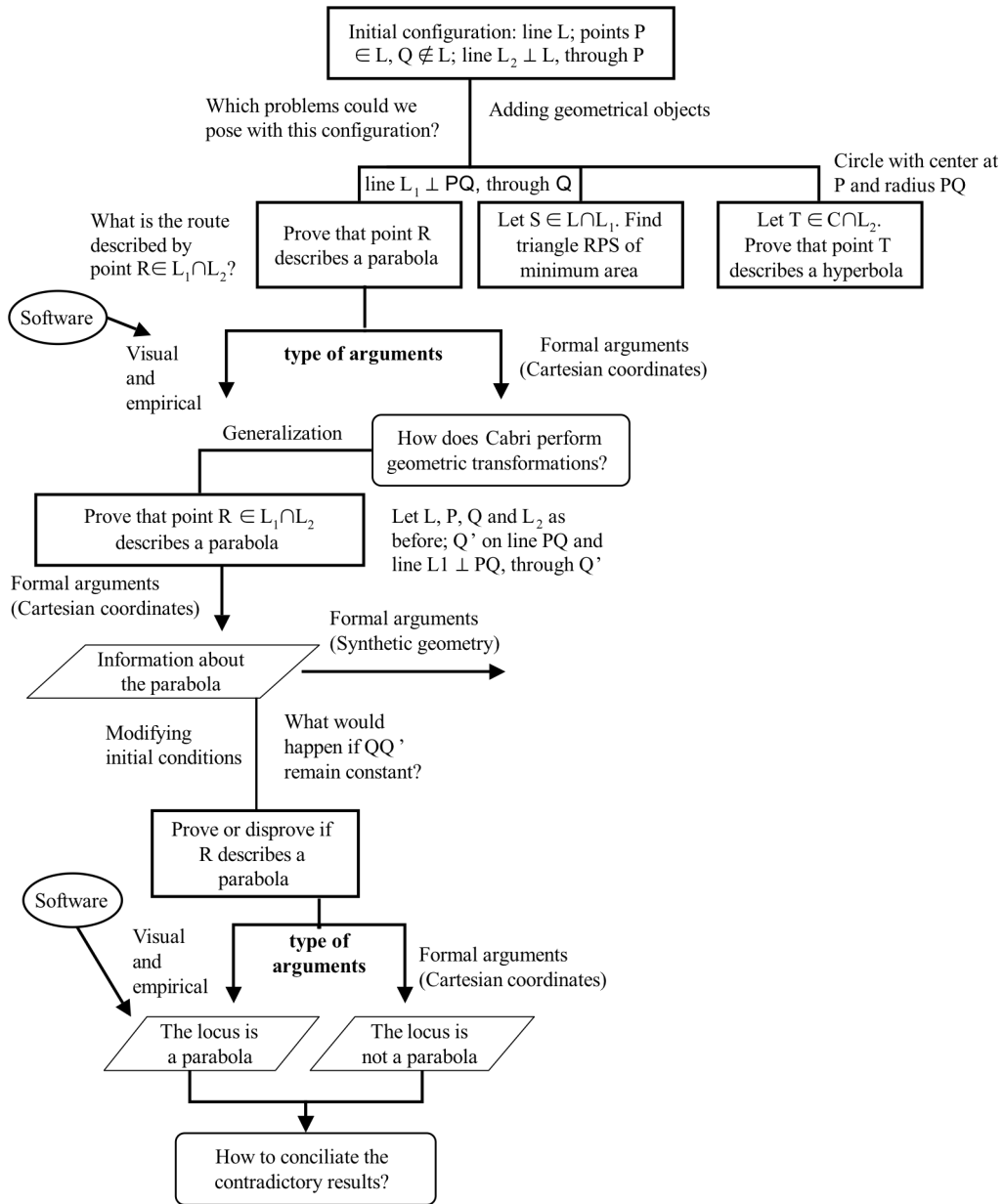


Figure 2: Hypothetical learning process in the context of a particular task.

At this stage, the dynamic representation of the task becomes a departure point to identify and explore diverse mathematical relations. Here, we document ways in which the working group explored the following general cases:

(a) Same assumptions on L , P and Q but now, it was taken an additional point Q' on the segment

PQ and the line L_1 that passes through the point Q' . What is the locus described by R when P moves along L ? How does the locus change when Q' moves along the segment PQ ?

An interesting part of the use of Cabri Geometry to formulate conjectures is that after proving the result the discussion group obtain more accurate information about the parabola. For example, knowing the focus and the directrix, the group could formulate the result in terms of synthetic geometry.

Let L be a line, Q a point not in L , $P \in L$, L_1 the line that passes through Q and P . Take a point $Q' \in L_1$ and draw the perpendicular line to L_1 that passes through Q' , calling it L_2 . Through P , Q' and Q draw perpendicular lines to L , calling these lines L_3 , L_4 and L_5 , respectively. Let T , S and R be the points of intersection of the lines L and L_4 ; L and L_5 ; L_2 y L_3 , respectively. Through Q' draw a perpendicular line to L_4 that intersects L_3 and L_5 at E and V respectively. Let F and W be points on L_5 such that $WV = VF = QS^2/4Q'T$. Let L_6 be the perpendicular to L_5 that passes through W and intersects L_3 at U . Then L_6 and F are the directrix and focus of a parabola with vertex at V .

Proof. The claim is equivalent to show that $UR=FR$. We have:

$$FR^2 = VE^2 + (UR - 2VF)^2 \dots(1)$$

From the similar triangles $PQ'T$ and PQS one has:

$$\frac{QS}{Q'T} = \frac{VE}{Q'E},$$

and from this, one obtains:

$$VE = \frac{SQ}{TQ'} Q'E.$$

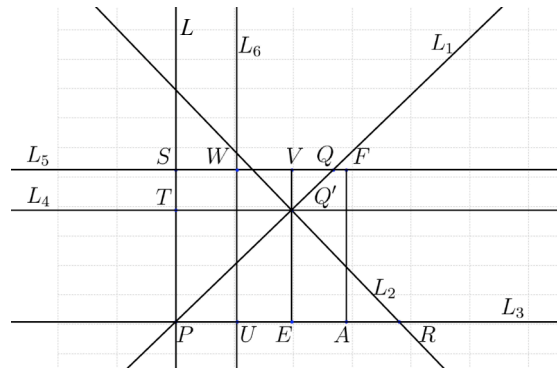


Figure 3: We have to prove that $UR = FR$.

Substituting the value of VF and VE in equation (1) and developing the binomial one arrives to:

$$\begin{aligned} FR^2 &= \frac{SQ^2 Q' E^2}{Q'T^2} + UR^2 - UR \frac{QS^2}{Q'T} + \frac{QS^4}{4Q'T^2} \\ &= UR^2 + \frac{SQ^2}{Q'T} \left(\frac{Q'E^2}{Q'T} - UR + FV \right). \end{aligned}$$

From the triangle $RQ'P$ we have $Q'E^2 = (PE)(ER)$; on the other hand $PE=Q'T$, hence from the previous equation one concludes that:

$$FR^2 = UR^2 + \frac{SQ^2}{Q'T} (ER - UR - FV).$$

We also have $ER-UR=-EU=-VW=-VF$; from which the conclusion follows.

In the last result, the group assumed that the second coordinate of the point Q' does not change; with this in mind some of the members of the group ask a very natural question. What would

happen if this condition is replaced by: the distance from Q to Q' remains constant?

(b) Assuming that L , P and Q are as above, but now the point Q' is the intersection of the line L' , passing through P and Q , and the circle C of radius r with center at Q . The lines L_1 and L_2 are constructed as before, and so is R . What is the locus described by R when P moves along L ? How does the locus behave when r approaches zero?

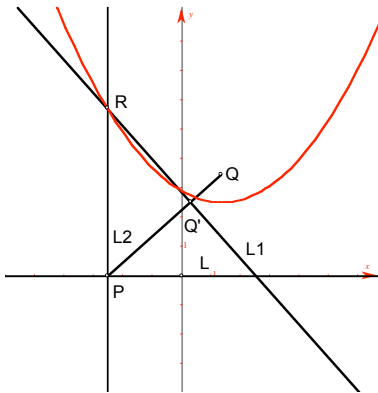


Figure 4: What is the locus of point R when point P moves along line L ?

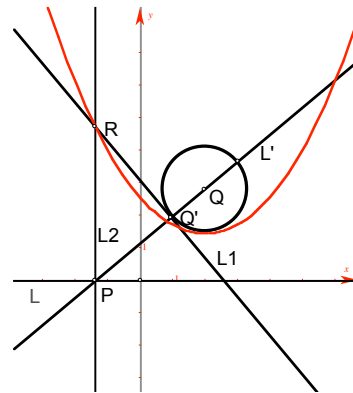


Figure 5: What is the locus of point R when point P moves along line L ?

In discussing part (b), with the use of Cabri Geometry the group has the chance to experiment and observe the behavior of the locus generated by R . One first approach shows results as shown in Figure 5, and it seems that the locus is a parabola, the *Equation tool* from Cabri Geometry even suggests that we are dealing with a parabola.

Nevertheless, taking a closer look at the geometrical behavior of the locus generated by R , there appears a graph as the one shown in Figure 6, which cannot be identified with the graph of a parabola. With this evidence, it is natural to ask for formal arguments to find out which kind of geometric object is described by point R . After performing calculations using a Coordinate System the group found that:

$$R = \left(x, \frac{1}{b} \left((x-a)^2 + b^2 \pm r \sqrt{(x-a)^2 + b^2} \right) \right),$$

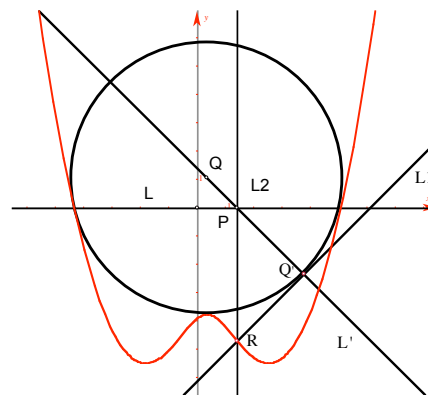


Figure 6: What locus is described by point R ?

where the center of the circle is (a, b) . It should be noticed that the second coordinate of R approaches $\left[(x-a)^2/b \right] + b$ when r approaches zero, which is the same result as (*), page 6. This result is consistent with the process of generalizing, an important aspect of the mathematical thinking.

Also the participants asked questions related with the way that Cabri performs geometric transformations. This led to think about the reliability of mathematical results obtained with the aid of a computer system. Here the group had the opportunity to point out the necessity of

analyzing the process and results obtain from the technological tool and ask questions related to the *axiomatic system* of it.

Closing remarks

Mathematical tasks are key elements of any professional development program that aims to revise and enhance teachers' mathematical and didactical knowledge. How should those tasks be discussed with teachers in order to identify explicitly ways of reasoning that are consistent with mathematical practice? We argue that tasks or problems should be addressed openly within an inquisitive community that promotes collaboration and mathematical reflection. In this process, the use of computational tools becomes relevant to represent some tasks dynamically and visualize diverse mathematical relations embedded in those tasks. It is evident that the conceptualization of the task as dilemmas, provide the opportunity to identify and explore relations, to open diverse lines of thinking or reflection that can lead the community or the problem solver to approach the task from diverse angles or perspectives. For example, the visual and empirical approach becomes important to identify relevant information, possible relations, and plausibility of solutions. The use of dynamic software offers the opportunity of utilizing particular heuristic strategies (searching for partial solutions) to solve the problem. Thinking of various approaches to the problem, another relevant problem solving activity, allows the problem solver to identify fundamental properties of the solution and possible relations or connections. Thus, problem solving is a continuous activity in which contents (from various domains), resources and strategies are used to initially construct a hypothetical learning trajectory that can be useful to orient and structure the practice of mathematical teachers. Finally, the group that worked on the task recognizes the relevance of approaching them within an inquisitive or inquiring community. The participants have developed a guide to implement the tasks. Of course, the plan and activities to implement the tasks in the professional development program were based on considering the trajectories that emerged during the group sessions.

An aspect, which is of crucial importance when using technological tools for solving mathematical problems, is related to providing support or formal arguments to results produced through the use of the tools. It is well accepted that technology is a powerful tool, however the results obtained should be examined rigorously in order to be accepted or rejected. In this respect Dick (2007, p. 1175) has introduced the term mathematical fidelity and has identify three areas in which a lack of mathematical fidelity can emerge: (i) mathematical syntax, (ii) underspecifications in mathematical structures and (iii) limitations in representing continuous phenomena with discrete structures and finite precision numerical computation. However these areas might not consider aspects related with reliability such as the results in the discussed example. We think that results that disagree with the expected ones has to do with the internal processing of the tool; related with this we suggest that a closer examination of the mathematical structure of the tool has to be done. At this respect our opinion agrees with Zbiek et al. (2007) whose statement is:

As technology becomes an increasingly integrated part of school mathematics, careful analysis of issues of mathematical fidelity [and reliability] will be needed. This type of research will necessitate intense collaboration involving mathematicians, computer scientist, and mathematics education researchers (p. 1176).

We also consider that this analysis should include categorizing levels of reliability of the tool. For instance we claim that the basic arithmetic operations (addition and multiplication within the precision range of calculators and computers) are 100% reliable. This is not the case for more sophisticated mathematical operations.

Acknowledgment

The first author acknowledges the support received from Conacyt through research project with reference #61996.

References

1. Adler, J., Ball, D., Krainer, K, Fou-Lai, L., & Novotna, J. (2005). Reflections on an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics*, 60, 359-381.
2. Cohen, A. (2001). Two reactions to the mathematical education of teachers. *Notices of the AMS*, 48(9), 985-991.
3. Davis, B. & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need) to know. *Educational Studies in Mathematics*, 61, 293-319.
4. Dick, T. (2007). Keeping the faith: Fidelity in technological tools for mathematics education. In G. W. Blume & M. K. Heid (Eds.), *Research on Technology and the Teaching and Learning of Mathematics: Syntheses, cases, and perspectives. Vol. 2: Cases and Perspectives* (pp. 333-339). Greenwich, CT: Information Age.
5. Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9(2), 187-211.
6. Lesh, R. & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 763-804). Charlotte, NC: Information Age.
7. NCTM (2000). *Principles and Standards for School Mathematics*. Reston, Virginia: National Council of Teachers of Mathematics.
8. Santos-Trigo, M. (2004). The role of dynamic software in the identification and construction of mathematical relationships. *Journal of Computers in Mathematics and Science Teaching*, 23 (4), 399-413.
9. Santos, M. (2007). *La resolución de problemas matemáticos. Fundamentos cognitivos*. México: Trillas.
10. Santos, M. et al. (2006). Constructing a Parabolas' World Using Dynamic Software to Explore Properties and Meanings. *International Journal for Technology in Mathematics Education*, 2 (3), 125-134.
11. Schoenfeld, A. (1994). Reflections on doing and teaching mathematics. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 53-70). Hillsdale, NJ: Lawrence Erlbaum.
12. Schoenfeld, A. (2000). Purposes and Methods of Research in Mathematics Education, *Notices of the AMS*, 47 (6), 641-649.
13. Simon, M. A. & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6(2), 91-104.
14. Zbiek, R. M. et al. (2007). Research on Technology in Mathematics Education. In F. K. Lester Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp.1169-1207). Charlotte, NC: Information Age.