Digital signals and systems
Unit impulse sequence

\[ \delta(n) = \begin{cases} 
1 & n = 0 \\
0 & n \neq 0 
\end{cases} \]
Unit step sequence

\[ u(n) = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0 
\end{cases} \]
Shifted sequences

\[ \delta(n-2) \]

\[ \delta(n+2) \]

\[ u(n-2) \]

\[ u(n+2) \]
Example

Given the following,

\[ x(n) = \delta(n + 1) + 0.5\delta(n - 1) + 2\delta(n - 2), \]

a. Sketch this sequence.
3.1. Sketch each of the following special digital sequences:
   a. $5\delta(n)$
   b. $-2\delta(n - 5)$
   c. $-5u(n)$
   d. $5u(n - 2)$

3.2. Calculate the first eight sample values and sketch each of the following sequences:
   a. $x(n) = 0.5^n u(n)$
   b. $x(n) = 5 \sin (0.2\pi n) u(n)$
   c. $x(n) = 5 \cos (0.1\pi n + 30^0) u(n)$
   d. $x(n) = 5(0.75)^n \sin (0.1\pi n) u(n)$

3.3. Sketch the following sequences:
   a. $x(n) = 3\delta(n + 2) - 0.5\delta(n) + 5\delta(n - 1) - 4\delta(n - 5)$
   b. $x(n) = \delta(n + 1) - 2\delta(n - 1) + 5u(n - 4)$
Linearity
Time invariance

\[ x_1(n) \]

\[ y_1(n) \]

\[ x_2(n) = x_1(n - n_0) \]

\[ y_2(n) = y_1(n - n_0) \]

\[ \text{Shifted by } n_0 \text{ samples} \]
A causal system is one in which the output $y(n)$ at time $n$ depends only on the current input $x(n)$ at time $n$, its past input sample values such as $x(n-1)$, $x(n-2)$, $\ldots$; Otherwise, if a system output depends on the future input values, such as $x(n+1)$, $x(n+2)$, $\ldots$, the system is noncausal.
3.6. Determine which of the following is a linear system.
   a. \( y(n) = 5x(n) + 2x^2(n) \)
   b. \( y(n) = x(n - 1) + 4x(n) \)
   c. \( y(n) = 4x^3(n - 1) - 2x(n) \)

3.7. Given the following linear systems, find which one is time invariant.
   a. \( y(n) = -5x(n - 10) \)
   b. \( y(n) = 4x(n^2) \)

3.8. Determine which of the following linear systems is causal.
   a. \( y(n) = 0.5x(n) + 100x(n - 2) - 20x(n - 10) \)
   b. \( y(n) = x(n + 4) + 0.5x(n) - 2x(n - 2) \)
A causal, linear, time-invariant system can be described by a difference equation having the following general form:

\[
y(n) + a_1 y(n - 1) + \ldots + a_N y(n - N) = b_0 x(n) + b_1 x(n - 1) + \ldots + b_M x(n - M),
\]

\[
y(n) = - \sum_{i=1}^{N} a_i y(n - i) + \sum_{j=0}^{M} b_j x(n - j).
\]
A linear time-invariant system can be completely described by its unit-impulse response, which is defined as the system response due to the impulse input $d(n)$ with zero initial conditions.

With the obtained unit-impulse response $h(n)$, we can represent the linear time-invariant system.
Example

Given the linear time-invariant system \( y(n) = 0.5x(n) + 0.25x(n-1) \) with an initial condition \( x(-1) = 0 \),

- Determine the unit-impulse response \( h(n) \).
- Draw the system block diagram.
- Write the output using the obtained impulse response.
Solution:

a. According to Figure 3.13, let \( x(n) = \delta(n) \), then

\[
h(n) = y(n) = 0.5x(n) + 0.25x(n - 1) = 0.5\delta(n) + 0.25\delta(n - 1).
\]

Thus, for this particular linear system, we have

\[
h(n) = \begin{cases} 
0.5 & n = 0 \\
0.25 & n = 1 \\
0 & \text{elsewhere}
\end{cases}
\]

b. The block diagram of the linear time-invariant system is shown as

![Block Diagram](image)

**FIGURE 3.15** The system block diagram in Example 3.7.

c. The system output can be rewritten as

\[
y(n) = h(0)x(n) + h(1)x(n - 1).
\]
Convolution (Digital convolution sum)

\[ y(n) = \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots \]

\[ h(n) = \ldots + h(-1)\delta(n+1) + h(0)\delta(n) + h(1)\delta(n-1) + h(2)\delta(n-2) + \ldots \]
Example

Given the difference equation
\[ y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0, \]
- Determine the unit-impulse response \( h(n) \).
- Draw the system block diagram.
- Write the output using the obtained impulse response.
- For a step input \( x(n) = u(n) \), verify and compare the output responses for the first three output samples using the difference equation and digital convolution sum.
Solution:

\[ h(n) = 0.25h(n-1) + \delta(n). \]

To solve for \( h(n) \), we evaluate

\[
\begin{align*}
  h(0) &= 0.25h(-1) + \delta(0) = 0.25 \times 0 + 1 = 1 \\
  h(1) &= 0.25h(0) + \delta(1) = 0.25 \times 1 + 0 = 0.25 \\
  h(2) &= 0.25h(1) + \delta(2) = 0.25 \times 0.5 + 0 = 0.0625 \\
  &\vdots
\end{align*}
\]

With the calculated results, we can predict the impulse response as

\[ h(n) = (0.25)^n u(n) = \delta(n) + 0.25\delta(n-1) + 0.0625\delta(n-2) + \ldots. \]

b. The system block diagram is given in Figure 3.16.
c. The output sequence is a sum of infinite terms expressed as

\[ y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots \]
\[ = x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots \]

d. From the difference equation and using the zero-initial condition, we have

\[ y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0 \]
\[ n = 0, \ y(0) = 0.25y(-1) + x(0) = u(0) = 1 \]
\[ n = 1, \ y(1) = 0.25y(0) + x(1) = 0.25 \times u(0) + u(1) = 1.25 \]
\[ n = 2, \ y(2) = 0.25y(1) + x(2) = 0.25 \times 1.25 + u(2) = 1.3125 \]

\\

Applying the convolution sum in Equation (3.15) yields

\[ y(n) = x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots \]
\[ n = 0, \ y(0) = x(0) + 0.25x(-1) + 0.0625x(-2) + \ldots \]
\[ = u(0) + 0.25 \times u(-1) + 0.125 \times u(-2) + \ldots = 1 \]
\[ n = 1, \ y(1) = x(1) + 0.25x(0) + 0.0625x(-1) + \ldots \]
\[ = u(1) + 0.25 \times u(0) + 0.125 \times u(-1) + \ldots = 1.25 \]
\[ n = 2, \ y(2) = x(2) + 0.25x(1) + 0.0625x(0) + \ldots \]
\[ = u(2) + 0.25 \times u(1) + 0.0625 \times u(0) + \ldots = 1.3125 \]
“... a linear time-invariant system can be represented by the convolution sum using its impulse response and input sequence.”
3.10. Find the unit-impulse response for each of the following linear systems.
   
   a. \( y(n) = 0.5x(n) - 0.5x(n - 2); \) for \( n \geq 0, \ x(-2) = 0, \ x(-1) = 0 \)
   
   b. \( y(n) = 0.75y(n - 1) + x(n); \) for \( n \geq 0, \ y(-1) = 0 \)
   
   c. \( y(n) = -0.8y(n - 1) + x(n - 1); \) for \( n \geq 0, \ x(-1) = 0, \ y(-1) = 0 \)

3.11. For each of the following linear systems, find the unit-impulse response, and draw the block diagram.

   a. \( y(n) = 5x(n - 10) \)
   
   b. \( y(n) = x(n) + 0.5x(n - 1) \)
A linear time-invariant system can be represented by using a digital convolution sum. Given a linear time-invariant system, we can determine its unit-impulse response $h(n)$, which relates the system input and output. (The sequences $h(k)$ and $x(k)$ in equations are interchangeable).

\[ y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \]

\[ = \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots \]

\[ y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \]

\[ = \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots \]
... para un sistema causal

\[ y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} x(k)h(n-k). \]
Methods to implement convolution

- Graphical (need reverse and shifted sequences)
- Formula
- Table

The reversed sequence is a mirror image of the original sequence, assuming the vertical axis as the mirror (If $h(n)$ is the given sequence, $h(-n)$ is the reversed sequence)
Given a sequence,

\[ h(k) = \begin{cases} 
3, & k = 0, 1 \\
1, & k = 2, 3 \\
0 & \text{elsewhere}
\end{cases} \]

where \( k \) is the time index or sample number,

a. Sketch the sequence \( h(k) \) and reversed sequence \( h(-k) \).

b. Sketch the shifted sequences \( h(-k+3) \) and \( h(-k-2) \).
Example

Using the following sequences defined in Figure 3.21, evaluate the digital convolution

\[ y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \]

a. By the graphical method.
b. By applying the formula directly.
c. using the table method
sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$
sum of product of $x(k)$ and $h(1-k)$: $y(1) = 1 \times 3 + 3 \times 2 = 9$
sum of product of $x(k)$ and $h(2-k)$: $y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$
sum of product of $x(k)$ and $h(3-k)$: $y(3) = 2 \times 2 + 1 \times 1 = 5$
sum of product of $x(k)$ and $h(4-k)$: $y(4) = 2 \times 1 = 2$
Animation

Animación 1

Animación 2
Formula method

\[
\begin{align*}
n = 0, \quad & y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9, \\
n = 1, \quad & y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9, \\
n = 2, \quad & y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11, \\
n = 3, \quad & y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5. \\
n = 4, \quad & y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2, \\
n \geq 5, \quad & y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.
\end{align*}
\]
Table method

<table>
<thead>
<tr>
<th>$k:$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(k):$</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h(-k):$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h(1-k)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h(2-k)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$h(3-k)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$h(4-k)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h(5-k)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$y(0) = 3 \times 3 = 9$

$y(1) = 3 \times 2 + 1 \times 3 = 9$

$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$

$y(3) = 1 \times 1 + 2 \times 2 = 5$

$y(4) = 2 \times 1 = 2$

$y(5) = 0$ (no overlap)
Convolution
>> h=[3 2 1];
>> x=[3 1 2];
>> conv(h,x)

ans =

      9      9     11      5      2

      . . .
A system representation using the unit-impulse response for the linear system

\[ y(n) = 0.25y(n-1) + x(n) \quad \text{for } n \geq 0 \text{ and } y(-1) = 0 \]

is determined in Example 3.8 as

\[ y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k), \]

where \( h(n) = (0.25)^n u(n) \). For a step input \( x(n) = u(n) \),

a. Determine the output response for the first three output samples using the table method.
\begin{array}{ccccccc}
\hline
k: & -2 & -1 & 0 & 1 & 2 & 3 & \ldots \\
x(k): & 1 & 1 & 1 & 1 & 1 & \ldots \\
h(-k): & 0.0625 & 0.25 & 1 & \quad & y(0) &= 1 \times 1 = 1 \\
h(1-k) & 0.0625 & 0.25 & 1 & \quad & y(1) &= 1 \times 0.25 + 1 \times 1 = 1.25 \\
h(2-k) & 0.0625 & 0.25 & 1 & \quad & y(2) &= 1 \times 0.0625 + 1 \times 0.25 + 1 \times 1 \\
& & & & & &= 1.3125 \\
\end{array}

Stop as required
3.15. Using the following sequence definitions,

\[ h(k) = \begin{cases} 
2, & k = 0, 1, 2 \\
1, & k = 3, 4 \\
0, & \text{elsewhere} 
\end{cases} \quad \text{and} \quad x(k) = \begin{cases} 
2, & k = 0 \\
1, & k = 1, 2 \\
0, & \text{elsewhere},
\end{cases} \]

evaluate the digital convolution

\[ y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n - k) \]

a. using the graphical method;
b. using the table method;
c. applying the convolution formula directly.
3.16. Using the sequence definitions

\[
x(k) = \begin{cases} 
-2, & k = 0, 1, 2 \\
1, & k = 3, 4 \\
0 & \text{elsewhere}
\end{cases}
\quad \text{and} \quad
h(k) = \begin{cases} 
2, & k = 0 \\
-1, & k = 1, 2 \\
0 & \text{elsewhere},
\end{cases}
\]

evaluate the digital convolution

\[
y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n - k)
\]

a. using the graphical method;
b. using the table method;
c. applying the convolution formula directly.
3.17. Convolve the following two rectangular sequences:

\[ x(n) = \begin{cases} 1 & n = 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1, 2 \\ 0 & \text{otherwise} \end{cases} \]

using the table method.
Convolution

Delta Function

Impulse Response

\[ \delta[n] \rightarrow \text{Linear System} \rightarrow h[n] \]
Examples convolution

a. Low-pass Filter

b. High-pass Filter
a. Inverting Attenuator

b. Discrete Derivative