Universidad Autónoma del Estado de Hidalgo
Instituto de Ciencias Básicas e Ingeniería
Centro de Investigación en Matemáticas
Licenciatura en Matemáticas Aplicadas

Inverse Problems

Víctor Nopal Coello
Advisor: Dr. Ricardo Cruz Castillo

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1. History

Jacques Hadamard

As an illustration of his idea, Hadamard uses the Laplace equation

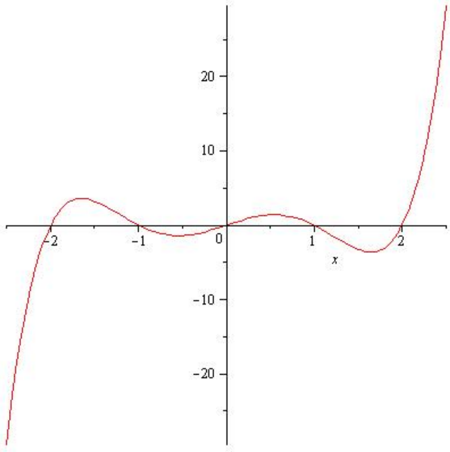
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

2. Inverse problems

In nature, as in mathematics, there are many pairs of problems where each of them is inverse to the other one, for instance:

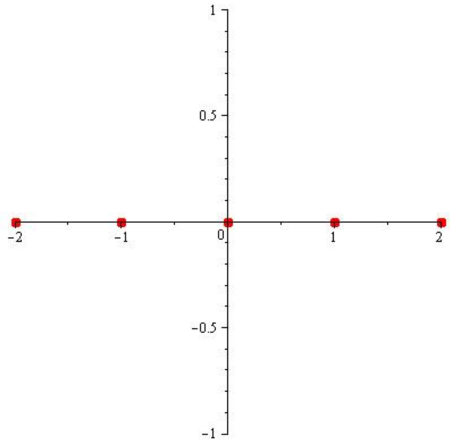
Direct problem: Find the zeros $x_1, \ldots, x_n$ of a given polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$

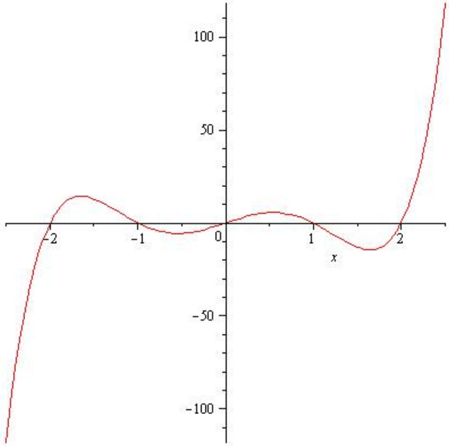
$$f(x) = x^5 - 5x^3 + 4x$$



Inverse Problem: Find a polynomial of degree n with given zeros $x_1, x_2, \cdots, x_n$.
Solution: $P(x) = c(x - z_1)(x - z_2)\cdots(x - z_n)$

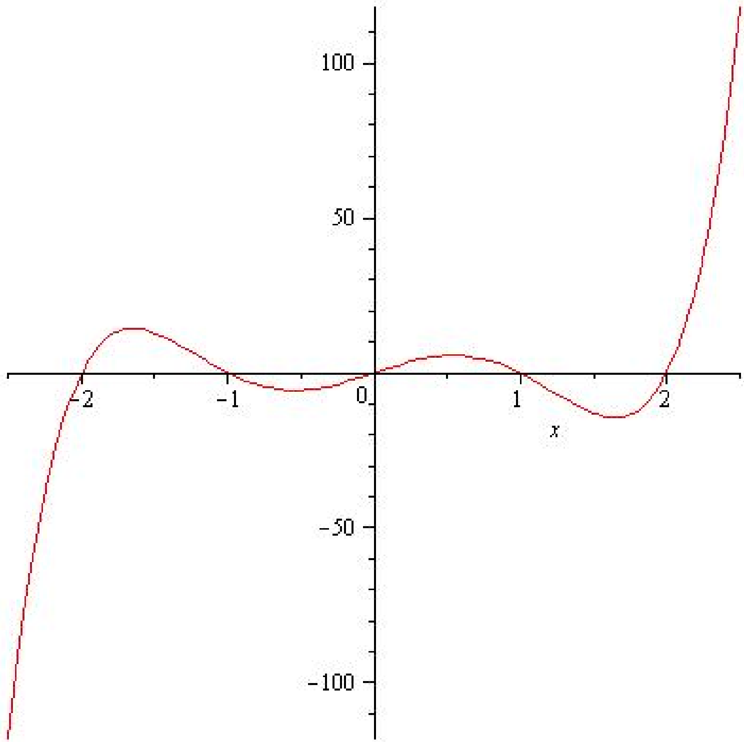
Example if $x = -2, -1, 0, 1, 2$





$p(x) = 4(x+2)(x+1)(x)(x-1)(x-2)$

Direct problem: Calculate the given polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ at given $x_1, \cdots, x_n$



$p(x) = 4x^5 - 20x^3 + 16x$

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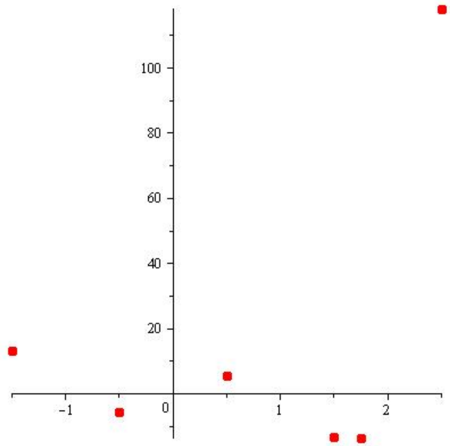
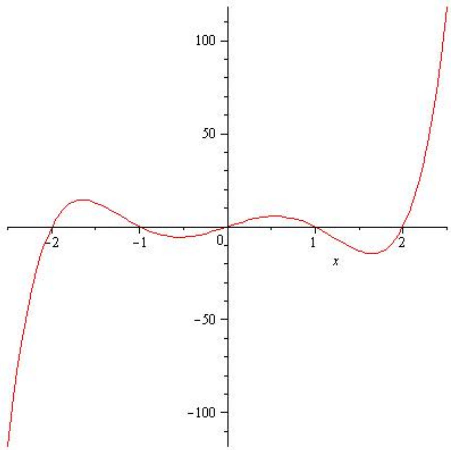
Inverse problem: Find a polynomial $P(x)$ of degree n that assumes given values $y_1, \cdots, y_{n+1} \in \mathbb{R}$ at given points $x_1, \cdots, x_{n+1} \in \mathbb{R}$.





Solution: Lagrange interpolation polynomial.

3. Well-posedness and ill-posedness

Andrei Nicolaevich Tikhonov

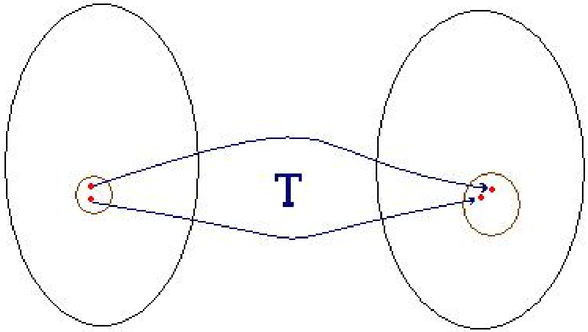
There exist two kinds of inverse problems: $Kx = y$
1.- The first kind is when we know y and the operator K, and we want determine x.
2.- The second kind is when we know y, x and some information about the operator K, and we want determine explicitly K.

Definition: Let X and Y be normed spaces and $K : X \to Y$ a (linear or nonlinear) mapping. The inverse problem of finding x, if y is known, such that $Kx = y$ is called properly posed or well-posed if the following holds.

1. Existence.
2. Uniqueness.
3. Stability.

Otherwise, the inverse problem ill-posed.

Stability



$T : X \to Y$.
If we have two points, x_1 and x_2, closes in X, then $T(x_1)$ and $T(x_2)$ are closes in Y

4. Example: well-posedness

Laplace Equation:

Suppose we have a disk of radius a and center in the origin where the temperature is independent of time. Suppose we know the temperature on the boundary of the disk. The inverse problem is determining the temperature distribution in the whole disk.

It is known that the temperature u satisfies the PDE

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the boundary condition

$$u(a, \theta) = f(\theta)$$

Definition: The functions that solve the Laplace equation are called harmonic functions.
We can easily see that if f and g are harmonic functions then $cf + g$ is a harmonic function too for $c \in \mathbb{R}$.

Maximum principles. In steady state the temperature cannot attain its maximum in the interior (unless the temperature is constant everywhere).

Analogously for the minimum.

Claim: This is a well-posed problem.

1) Existence: It is known that the solution is

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$$

$0 \le r < a$, $-\pi \le \theta < \pi$
Where

$$f(\theta) = u(a,\theta) = \sum_{n=0}^{\infty} A_n a^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n a^n \sin(n\theta)$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

$$A_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta$$

$$B_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta$$

2) Uniqueness: Suppose that we have two solutions u_1 and u_2 for the Laplace equation with the same boundary condition, so $u_1 - u_2$ is a harmonic function in the disk and it satisfies the homogeneous boundary condition $u_1 - u_2 = 0$. Maximum principle implies that $u_1 - u_2 = 0$ everywhere inside the disk, therefore $u_1 = u_2$.

3) Stability: If we change f, for f_1, such that $\|f - f_1\|_\infty < \epsilon$, that is $\max\{|f(x) - f_1(x)|\} < \epsilon$, then we get a new solution u_1 for the Laplace equation with the condition $u_1 = f_1$ in the boundary of the disk. Since u and u_1 are harmonic functions then $u - u_1$ is a harmonic function too and therefore its maximum and minimum are in the boundary, then $\|u - u_1\|_\infty < \epsilon$.

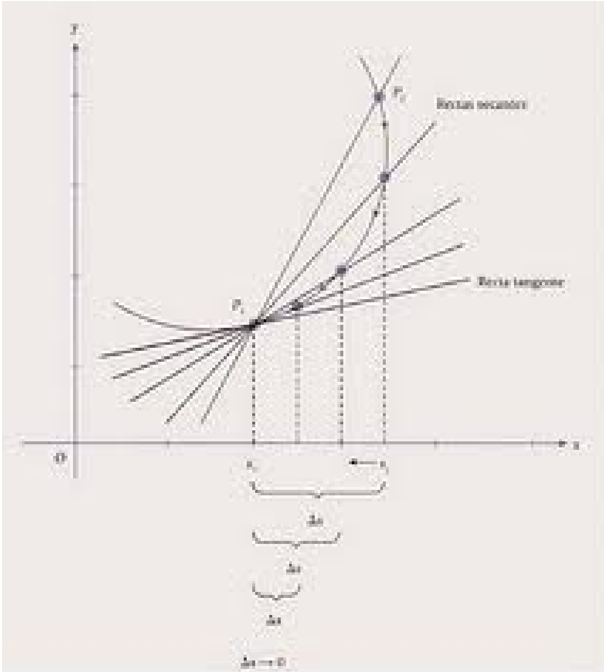
5. Example: ill-posedness

Differentiation

The direct problem is to find the antiderivative y with $y(0) = 0$ of a given continuous function x on $[0, 1]$, i.e., compute

$$y(t) = \int_0^t x(s)ds,\ t \in [0, 1].$$

In the inverse problem, we are given a continuously differentiable function y on $[0, 1]$ with $y(0) = 0$ and we want to determinate $x = y'$.



This means we have to solve the integral equation $Kx = y$, where $K : X = C[0,1] \to Y = C[0,1]$ is defined by

$$(Kx)(t) := \int_0^t x(s)ds, \qquad t \in [0,1], \text{ for } x \in C[0,1]$$

Here we equip X with the norm $\|\cdot\|_\infty$

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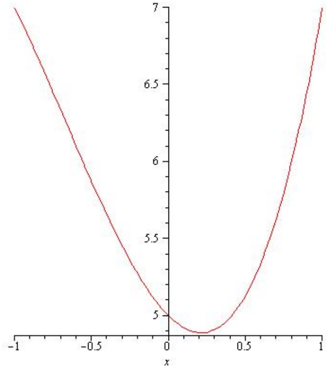
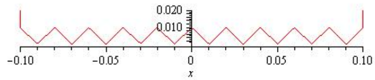
Claim: The inverse problem of differentiation is ill-posed.

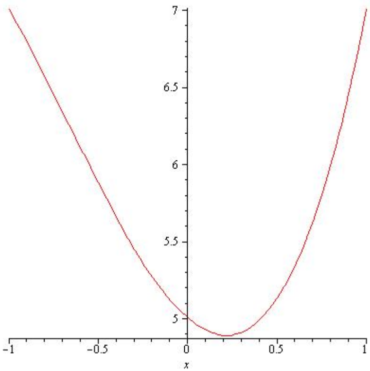
1)When we perturb the function y then the resulting function $\tilde{y}$ doesn't have to be differentiable.



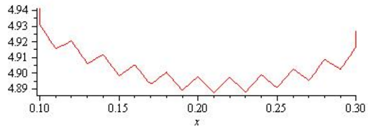
y



$g(x)$



$\tilde{y} = y + g$



Zoom($\tilde{y}$)

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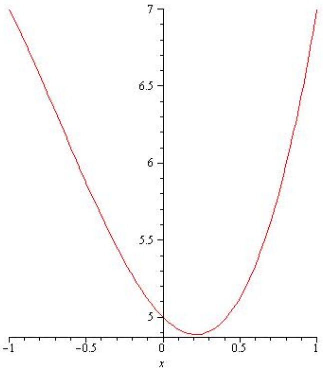
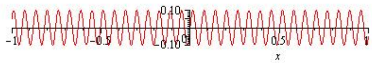
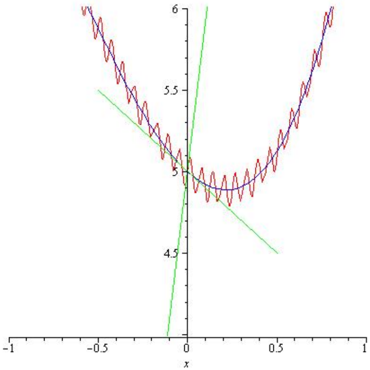
2)Even though $\tilde{y}$ is differentiable, the derivative at a point x doesn't have to be close to the derivative of y at the same point.



y



$\delta \sin(x/\delta^2)$



$\tilde{y} = y + \delta \sin(x/\delta^2)$

If δ is very small, then the error in the solution is huge.

What happen if we change the norm?

Now we define $Y := \{y \in C^1[0,1] : y(0) = 0\}$ equipped with the norm $\|x\|_{C^1} := \max\limits_{0 \le t \le 1} |x'(t)|$.

If we perturb y by a function $f(t)$, i.e., $\tilde{y}(t) = y(t) + f(t)$, and we suppose $f(t) = \int_0^t g(s)ds$.

It can be shown that the error in the solution is lesser than $\|f(t)\|_\infty$.

Therefore if $\|f\|_\infty$ is small then the error in the solution is small too.

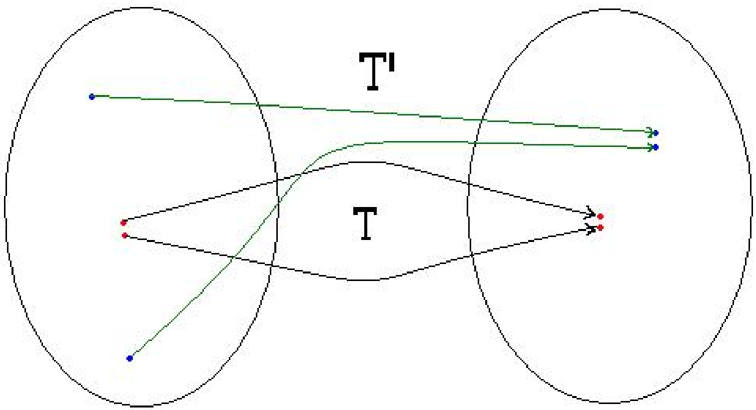
Example's conclusions:

1) The existence and the uniqueness depend only on the operator K and on the spaces X y Y.

2) The stability also depends of the norm in the spaces X y Y.

6. The worst case error

Compact Operator



$T : X \to Y$ is a continous operator and $T' : X \to Y$ is a compact operator.

Theorem
Let X and Y be normed spaces and $K : X \to Y$
a linear compact and one-to-one operator. Let the dimension of X be infinite. Then there exist a sequence $(x_n) \subset X$ such that $Kx_n \to 0$ but (x_n) does not converge. We can even choose (x_n) such that $\|x_n\| \to \infty$. Furthermore $K^{-1} : Y \supset K(X) \to X$ is unbounded.

Observations:

1) It can be seen that if the operator K, between spaces with infinite dimension, is compact then the inverse problem $Kx = y$ is ill-posed.
2) It is known that the integral operators like $(Kx)(t) := \int_0^t x(s)ds$ are compact.

How large could the error be in the worst case if the error in the right side y is at most δ?
The answer is already given by the previous theorem: If the errors are measured in norms such that the integral operator is compact, then the solution error could be arbitrarily large.

Lanczos wrote in his book "*Linear Differential Operators*" that *A lack of information cannot be remedied by any mathematical trickery!*

Let y and $\tilde{y}$ be twice continuously differentiable and let a number $E > 0$ be available with

$$\|y''\|_\infty \leq E \text{ and } \|\tilde{y}''\|_\infty \leq E$$

Set $z := \tilde{y} - y$, and assume that $z'(0) = z(0) = 0$ and $z'(t) \geq 0$ for $t \in [0, 1]$.
It can be proved that $\|\tilde{x}(t) - x(t)\| \leq 2\sqrt{Ez(t)}$

Definition: Let $K : X \to Y$ be a linear bounded operator between Banach spaces, $X_1 \subset X$ un subspace, and $\|\cdot\|_1$ a "stronger" norm on X_1, i.e, there exists $c > 0$ such that $\|x\| \leq c\|x\|_1$ for all $x \in X_1$. Then we define

$$\mathcal{F}(\delta, E, \|\cdot\|_1) := \sup\{\|x\| : x \in X_1, \|Kx\| \leq \delta, \|x\|_1 \leq E\}$$

and call $\mathcal{F}(\delta, E, \|\cdot\|_1)$ *the worst-case error*, for the error δ in the date and a priori information $\|x\|_1 \leq E$.

Definitions:

1)

$$L^2(I) := \{x : I \to \mathbb{R} : \int_I (x(s))^2 ds < \infty\}$$

If $x \in L^2(I)$ then $\|x\|_{L^2} = \sqrt{\int_I (x(s))^2 ds}$

2)

$$H^p(a,b) := \{x \in C^{p-1}[a,b] : x^{(p-1)}(t) = \alpha + \int_a^t \psi ds\}.$$

Where $\alpha \in \mathbb{R}$ and $\psi \in L^2(a,b)$

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Again, consider the problem of differentiation, but now set $X = Y = L^2(0,1)$,

$$(Kx)(t) := \int_0^t x(s)ds, \qquad t \in (0,1),\ x \in L^2(0,1)$$

and

$$X_1 := \{x \in H^1(0,1) : x(1) = 0\}$$

If $x \in X_1$ then $x(t) = \alpha + \int_a^t \psi ds$

We define $\|x\|_1 := \|x'\|_{L^2}$. Then the norm $\|\cdot\|_1$ is stronger than $\|\cdot\|_{L^2}$. We can prove for every $E > 0$ and $\delta > 0$

$$\mathcal{F}(\delta, E, \|\cdot\|_1) \leq \sqrt{\delta E}$$

Now if we define

$$X_2 := \{x \in H^2(0,1) : x(1) = 0, x'(0) = 0\}$$

If $x \in X_2$ then $x'(t) = \alpha + \int_a^t \psi ds$

And $\|x\|_2 := \|x''\|_{L^2}$. Then the norm $\|\cdot\|_2$ is stronger than $\|\cdot\|_{L^2}$. We can prove for every $E > 0$ and $\delta > 0$

$$\mathcal{F}(\delta, E, \|\cdot\|_2) \leq \delta^{2/3} E^{1/3}$$

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