Matrix Commutativity

Admissible Patterns

Aguayo, Guillén, Kraft, Mason, Romero ‡

California State University Channel Islands

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   An **admissible** pattern is a pattern of specified entries of X which can be completed to commute with A.

The Omega matrix and the Psi matrix

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Example

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 8 & 3 \\ 2 & 0 & 6 \end{bmatrix}$$

Example: $\Omega(A)$

$$\text{Then } \Omega(A) = \begin{bmatrix} 0 & 0 & 5 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 7 & 3 & 0 & 0 & 0 & 0 & -2 & 0 \\ 2 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -7 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 \\ -5 & 0 & 0 & -3 & 0 & 0 & -5 & 0 & 5 \\ 0 & -5 & 0 & 0 & -3 & 0 & 0 & 2 & 3 \\ 0 & 0 & -5 & 0 & 0 & -3 & 2 & 0 & 0 \end{bmatrix}$$

Example: Basis for Nullspace

And a basis for the nullspace of $\Omega(A)$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & {}^{2}/_{3} & {}^{5}/_{2} \\ 0 & 0 & 0 \\ 0 & 0 & {}^{5}/_{2} \\ 0 & {}^{7}/_{2} & {}^{3}/_{2} \\ 1 & 0 & {}^{5}/_{2} \end{bmatrix}$$

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Using the Vec ordering:

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$$\text{Rank } \Omega(A) = \text{Rank } \Omega(A_{\alpha c})$$

The easiest case

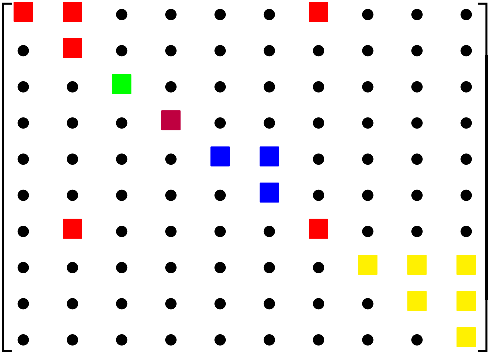
Let A be a Matrix of size $n \times n$ and not all the diagonal entries are equal (some can be equal), a pattern on X is the diagonal (it has n elements)

$$A1 = \begin{bmatrix} a_1 & a_{n+1} & \cdots & a_{n(n-1)+1} \\ a_2 & a_1 & & \vdots \\ \vdots & \vdots & \ddots & \\ a_n & a_{2n} & \cdots & a_i \end{bmatrix}$$

$$X1 = \begin{bmatrix} \color{red}{x_1} & x_{n+1} & \cdots & x_{n(n-1)+1} \\ x_2 & \color{red}{x_{n+1}} & & \vdots \\ \vdots & \vdots & \color{red}{\ddots} & \\ x_n & x_{2n} & \cdots & \color{red}{x_{n^2}} \end{bmatrix}$$

General Case

Let A be any Matrix of size $n \times n$, we can rewrite $AX = XA$ as $JY - YJ = 0$. We know the patterns for any matrix J, with multiple Jordan blocks having any kind of eigenvalues. A pattern for A is the same pattern for J:



General answer

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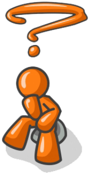
General answer

- The way from $AX = XA$ to $JY - YJ = 0$:
- For all square matrix A we can get its Jordan Canonical Form:
- $$A = Q^{-1}JQ$$
- And we change the matrix X by $Y = QXQ^{-1}$
- We can complete any matrix A as the same form as we can complete its matrix J, except for the easiest case.

Open Questions



- Does Y give us more information about maximal admissible pattern for A?

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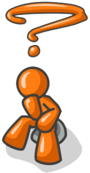

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- How can we order the matrix J?

Elements of the Symmetric Group

- $n \times n$ Matrices with a single 1 in each row and column; every other entry is zero

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- The group operation is matrix multiplication

Properties of the Group

- Interesting because it's nonabelian except for $n \in \{1, 2\}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
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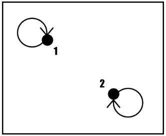
- Sadly, it's closed under neither matrix addition nor scalar multiplication.
- We'll examine the commutativity equation, $AX - XA = 0$ where A is a fully specified permutation matrix, and X is a partially specified matrix which we must complete as a permutation matrix.

Graphs and Orbits

We can rewrite a permutation matrix as a digraph. Notice the orbits.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim$$



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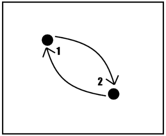

Powers and Commutativity

- Powers of any matrix will commute with it. Permutation matrices are no different.
- But that doesn't exhaust commutors.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

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- Neither power is a matrix of the other.

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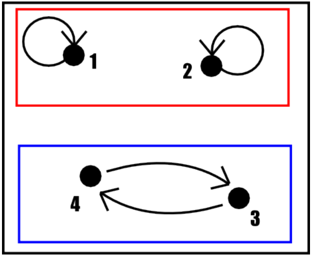

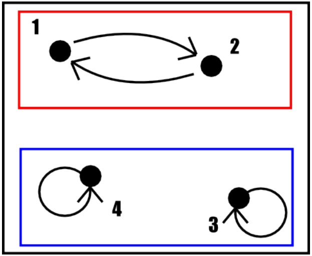

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Admissible Patterns for $A = Q^{-1}JQ$

- An admissible pattern for a Jordan block is generally of the form: $\alpha = \{(a_1, a_1), (a_2, (a_2, a_2 + 1), \cdots, (a_{n-1}, a_{n-1} + (n-2)), (1, n)\}$ for $1 \leq a_i \leq n - i + 1$

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- Each element in an admissble pattern β for a $n \times n$ matrix A that is permutation equivalent to J is of the form $(\sigma(i), \sigma(j))$ for each $(i, j) \in \alpha$. σ is the permutation associated with Q^{-1}.

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- Each element in an admissble pattern β for a $n \times n$ matrix A that is permutation equivalent to J is of the form $(\sigma(i), \sigma(j))$ for each $(i, j) \in \alpha$. σ is the permutation associated with Q^{-1}.
- If $A = Q^{-1}JQ$, then $\beta = \{(\sigma(a_1), \sigma(a_1)), (\sigma(a_2), \sigma(a_2 + 1)), ..., (\sigma(a_n), \sigma(a_n + (n-1)))\}$ is an admissible pattern for the matrix A.

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- New Question: Suppose we have a pattern β, then can we find a matrix A that is permutation equivalent to a Jordan block such that β is an admissible pattern for A?

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- We try to find the σ^{-1} that makes this true because we can then easily get Q^{-1}.
- Answer: If we can find Q^{-1}, then we can find the $n \times n$ matrix $A = Q^{-1}JQ$ for which β is admissible for.

Finding a Matrix for a given β Example

- Suppose $\beta = \{(1,1),(3,1),(3,2)\}$. Is there an A such that β is an admissible pattern for it?

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- β-partial matrix $\rightarrow$ α-partial matrix

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  X-Unspecified entries which can be ANYTHING, $\square$-Specified entries
- $\alpha = \{(2,2),(1,2),(1,3)\}$

Example continued

- There is a permutation Q that transforms the β-partial matrix into the α-partial matrix.
  $Q(\beta\text{-partial matrix})Q^{-1} = \alpha\text{-partial matrix}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \color{red}{\square} & X & X \\ X & X & X \\ \color{blue}{\square} & \color{green}{\square} & X \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} X & \color{blue}{\square} & \color{green}{\square} \\ X & \color{red}{\square} & X \\ X & X & X \end{bmatrix}$$

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- Hence, $A = Q^{-1}JQ = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 1 & 0 & \lambda \end{bmatrix}$ is a matrix that β is admissible for!

Graph Theoretic Approach

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- $n\binom{n^2-n}{n-1}$
- Worked on a 4×4 matrix $\Rightarrow$ 880 different graphs
- Some patterns, when permuted, can be completed to commute with a Jordan block.
- What other patterns can we complete to commute with other non-derogatory matrices?

N-cycle Matrices

- Some patterns can be completed to commute with an "N-cycle" matrix.

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Example

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- For each specified entry a_{ij}, $(i-j)$ mod 4, is distinct.

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- Ongoing work will attempt to give complete answers to these questions.

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