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Traveling wave solutions in 1d degenerate parabolic lattices

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- Goal: advance current understanding of nonlinear diffusion in spatially discrete systems.

- Specific objective: systematic study of semidiscrete models of 1d PME,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad m > 1 .$$

Work in progress.

Contents

Part I: discrete models

1. Porous medium equation (PME)

$$\frac{\partial u}{\partial t} = \Delta(u^m), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad m > 1.$$

Physical applications: flow of an isentropic gas through a porous medium, groundwater filtration, heat radiation of plasmas, spread of a thin layer of viscous fluid under gravity, boundary layer theory, population dynamics, etc. (cf. [Váz07], [Aro86], [GM77]).

Will focus on 1d PME: $$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad m > 1.$$

1. Porous medium equation (PME)

Important quantities associated to PME:

scaled pressure: $w = \frac{m}{m-1} u^{m-1}$ satisfies

$$\frac{\partial w}{\partial t} = (m-1)\, w \frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad m > 1. \qquad \text{(SPE)}$$

M-pressure: $v = u^m$ satisfies

$$\frac{\partial v}{\partial t} = m v^{\frac{m-1}{m}} \frac{\partial^2 v}{\partial x^2}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad m > 1. \qquad \text{(mPE)}$$

The theory of the PME can alternatively be developed from (SPE).

1. Semidiscrete models

(a) Discrete scaled pressure (DSP) $w_j := w(jh), \quad h>0, \quad j \in \mathbb{Z}$

$$w_x(jh) \quad \rightarrow \quad (w_{j+1} - w_{y-1})/2\mathrm{h}$$
$$w_{xx}(jh) \quad \rightarrow \quad (w_{j+1} - 2\mathrm{w}_j + w_{j-1})/h^2$$

Let $(W_t)(j) := w_j(t), \quad j \in \mathbb{Z}$ (DSP)

then $$\boxed{\dot{w}_j = \alpha(m-1)w_j(w_{j+1} - 2\mathrm{w}_j + w_{j-1}) + \frac{\alpha}{4}(w_{j+1} - w_{j-1})^2, \quad j \in \mathbb{Z}, \quad \alpha = h^{-2}}$$ (DSPE)

Define discrete scaled density (DSD):

$$(U_t)(j) = u_j(t) := \beta(w_j(t))^{\frac{1}{m-1}}, \quad j \in \mathbb{Z}, \quad \beta = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}}$$ (DSD')

then $$\dot{u}_j = \alpha\gamma(m-1)u_j(u_{j+1}^{m-1} - 2\mathrm{u}_j^{m-1} + u_{j-1}^{m-1}) + \frac{\alpha\gamma}{4}u_j^{2-m}(u_{j+1}^{m-1} - u_{j-1}^{m-1})^2, \quad j \in \mathbb{Z}$$ (DPME')

where $\gamma = m(m-1)^{-2}$

1. Semidiscrete models

(b) Discrete m-pressure (DM-P) $(V_t)(j) := v_j(t), \quad j \in \mathbb{Z}$ (DM-P)

$$\boxed{\dot{v}_j = \alpha m v_j^{\frac{m-1}{m}} (v_{j+1} - 2\mathrm{v}_j + v_{j-1}), \quad j \in \mathbb{Z}} \qquad \text{(DM-PE)}$$

(c) Discrete scaled density (DSD)

Let $G(x)$ sufficiently smooth, then

$$G(x) = G(x_j) + \partial_x G(x_j)(x - x_j) + \frac{1}{2}\partial_x^2 G(x_j)(x - x_j)^2 + O((x - x_j)^3).$$

Let $x_{j+1} - x_j = x_j - x_j - 1 = h > 0$ then it follows from above that

$$G(x_{j+1}) + G(x_{j-1}) = 2\mathrm{G}(x_j) + \partial_x{}^2 G(x_j) h^2 + O(h^4).$$

If $G(x) = F(u(x))$ and $F = u^m$ we get $\partial_x^2(u^m)(x_j) = \frac{1}{h^2}(u^m(x_{j+1}) - 2\mathrm{u}^m(x_j) + u^m(x_{j-1})) + O(h^2)$

1. Semidiscrete models

This suggests defining $(U_t)(j) := u_j(t)\,,\quad j \in \mathbb{Z}$ (DSD)

such that $\boxed{\dot{u}_j = \alpha(u_{j+1}^m - 2u_j^m + u_{j-1}^m)\,,\quad j \in \mathbb{Z}\,,\quad \alpha = h^{-2}}$ (DPME) "classical discretization"

Lemma: *when dealing with nonnegative solutions,* (DM-PE) *and* (DPME) *are equivalent; i.e., if* (u_j) *satisfies* (DPME) *then* $(v_j = u_j^m)$ *satisfies* (DM-PE)*; likewise, if* (v_j) *satisfies* (DM-PE) *then* $(u_j = v_j^{1/m})$ *satisfies* (DPME).

Issues concerning (DPME): numerical (failure to reproduce simultaneous and non-simultaneous Blow-up conditions, cf. [BQR05]), there is more than one way of discretizing PME as opposed to just one in the case of (SPE)

Questions:
Can we find other semidiscrete models for PME which do not suffer from numerical drawbacks like its "classical" discretization (e.g.: (DPME')). What can we learn from such modes (e.g., existence proofs of traveling waves and diffusion phenomena)?

Semidiscrete models for

$$(u^m)_{xx} = m(m-1)u^{m-2}(u_x)^2 + m\,u^{m-1}u_{xx}$$

$m(m-1)\times$	$\times$	$+\,m\times$	$\times\,\alpha\left[u_{j+1} - 2u_j + u_{j-1}\right]$ three-point difference
single-power term u_j^{m-2}	symmetric product sum $\dfrac{\alpha}{3}\left[(u_{j+1}-u_j)^2 + (u_{j+1}-u_j)(u_j-u_{j-1}) + (u_j-u_{j-1})^2\right]$	u_j^{m-1}	
symmetric-product average $\dfrac{1}{m-1}\sum_{k=0}^{m-2} u_{j+1}^{m-2-k}u_{j-1}^{k}$	square average $\dfrac{\alpha}{2}\left[(u_{j+1}-u_j)^2 + (u_j-u_{j-1})^2\right]$	$\dfrac{1}{m}\sum_{k=0}^{m-1} u_{j+1}^{m-1-k}u_{j-1}^{k}$	
$(m-2=2s+1)$ $\dfrac{u_j^{s+1}}{s+1}\sum_{k=0}^{s} u_{j+1}^{s-k}u_{j-1}^{k}$ odd symmetric-product avg.	secant approximation $\dfrac{\alpha}{4}(u_{j+1}-u_{j-1})^2$	$(m-2=2s+1)$ $\dfrac{1}{2s+3}\sum_{k=0}^{2(s+1)} u_{j+1}^{2(s+1)-k}u_{j-1}^{k}$	
$(m-2=2s)$ $\dfrac{1}{2s+1}\sum_{k=0}^{2s} u_{j+1}^{2s-k}u_{j-1}^{k}$		$(m-2=2s)$ $\dfrac{u_j^{s+1}}{s+1}\sum_{k=0}^{s} u_{j+1}^{s-k}u_{j-1}^{k}$	

Table *A:* col. 1 and 3 entries correspond. 12 models total

Part II: traveling wave solutions in 1d reaction-diffusion lattices

3. Traveling wavefronts

Prototype lattice, *discrete cable equation* (bistable): electrical activity in myelinated nerve fibers,

$$\dot{u}_j = \alpha(u_{j+1} - 2u_j + u_{j-1}) + f(u_j), \quad j \in \mathbb{Z}$$

where $f(u) = u(u-1)(a-u)$, or $f(u) = -u + H(u-a)$; $0 < a < 1$ (cf. [KS98]).

Traveling wave with speed *c*: $u_j(t) = \phi(j+ct)$, $\forall\ j \in \mathbb{Z}$, $\forall\ t \in \mathbb{R}$

such that $\phi: \mathbb{R} \to [0,1]$, $\phi \in C^1$, $\lim_{\xi \to \infty} \phi(\xi) = 0$, $\lim_{\xi \to \infty} \phi(\xi) = 1$

Theorem (Keener, 1987): *for any bistable function f, there is a number α^* such that if $\alpha \le \alpha^*$ then the discrete bistable equation has a standing solution, i.e. a solution to*

$$0 = \alpha(u_{j+1} - 2u_j + u_{j-1}) + f(u_j)$$

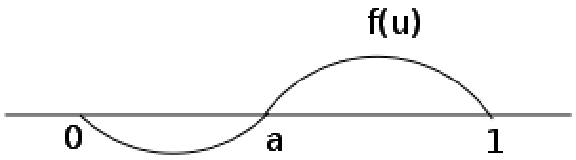
and therefore propagation fails.

(On proof: maximum principle and comparison arguments.)

3. Traveling wavefronts

Some results.

Theorem (Zinner, 1992): *discrete Nagumo eq.* $\dot{u}_j = \alpha(u_{j+1} - 2u_j + u_{j-1}) + f(u_j)$
f Lipschitz continuous and such that



and $\int_0^1 f(x)\,dx > 0$.

Then there exists d^* *such that for* $d > d^*$ *DN eq admits a traveling wave solution, with monotone increasing differentiable profile, which propagates at constant speed* c > 0; *i.e.,*

$$U \in C^1(\mathbb{R}, (0,1)), \quad U(-\infty) = 0, \quad U(\infty) = 1, \quad U'(x) > 0 \;\; \forall x \in \mathbb{R}$$

On proof: (artisan) Brower's fixed point and a homotopy invariance arguments.

3. Traveling wavefronts

Zinner's proof structure has 4 steps:

Step 1: consider auxiliary system:

$$\dot{v}_j = \alpha(u_{j+1} - 2u_j + u_{j-1}) + u_j - \frac{1}{4}$$
$$u_j = P(v_j)$$

Where $P(v_j) := \begin{cases} 0 & if \quad v_j < 0 \\ v_j & if \quad 0 \le v_j \le 1 \\ 1 & if \quad 1 < v_j \end{cases}$

Auxiliary system has a monotone traveling wave solution only if finitely many $u_j(0)$ are different from zero or one; therefore, can consider system is finite dimensional.

3. Traveling wavefronts

Step 2: set up a fixed point problem for the initial value problem,

$$\dot{v}_j = \alpha(u_{j+1} - 2u_j + u_{j-1}) + u_j - \frac{1}{4}$$
$$u_j = P(v_j)$$
$$v_j(0) = x_j; \quad 0 \le x_j \le 1, \quad j = 0, \dots N; \quad u_{-1} = 0, u_{N+1} = 1$$

ivp has a unique solution which depends continuously on the initial data $u(x;t) = \{u_j(x;t)\}_{j=0}^{N}$

In a suitably chosen (nonempty and convex) space X of increasing sequences $\{x_j\}_{j=0}^{N}$ the following "shifted" Poincaré map is continuous and maps $\bar{X}$ into X

$$T : \bar{X} \to \mathbb{R}^{N+1}$$
$$(Tx)_j := \begin{cases} 0 & \textit{for} \quad j = 0 \\ u_{j-1}(x;\tau) & \textit{for} \quad j = 1, \dots, N \end{cases}$$

By Brower's fixed point theorem, *T* has a fixed point.

3. Traveling wavefronts

Step 3: fixed point of $T_h = T_0$ is a traveling wave for the auxiliary problem.
Homotopy argument: consider sequence $\{h_k\}$ converging to f, continuously deform h_k into h_{k+1} so that fixed points of T_k are continued to fixed points of T_{k+1}.

Step 4: the (shifted!) sequence of fixed points $\{u^{(k)}\}$ converges to a fixed point of DN eq.

Zinner 1991: Global stability of traveling waves (*f* can have more than one zero in (0,1)).
Zinner et al (1993): traveling waves for the discrete Fisher equation *f(0)=f(1)=0, f(x)>0* in (0,1).

3. Traveling wavefronts

Fu et al (1999): existence of traveling wavefronts for

$$\dot{u}_j = \alpha (u_{j+1}^m - 2u_j^m + u_{j-1}^m) + f(u_j), \quad j \in \mathbb{Z}, \quad m \geq 1$$
$$f(u) = u(1-u)$$

m >1: DPME with a Fisher-type reaction term.

*"Novelty:"** introduce Monotone Iteration Method (MIM) for $m \geq 2$, extending Zinner's case.

Drawbacks: method doesn't work for *1<m<2* (but Zinner's argument does), MIM uses the explicit form of *f.*

*the concept of upper and subsolution, pivotal for MIM, appears already in [Zin93]

3. Traveling wavefronts

MIM steps:

Step 1: choose ansatz form,

$$u_j(t) = \phi(j + ct) \quad \forall n \in \mathbb{Z}, \quad \forall t \in \mathbb{R},$$

$$\phi : \mathbb{R} \to [0,1], \ \phi(-\infty) = 0, \ \phi(+\infty) = 1$$
$$c > 0, \ \phi \in C^1$$

3. Traveling wavefronts

Step 2: substitute in DPM eq:

$$\boxed{c\,\phi'(\xi)=d\,[\phi^m(\xi+1)-2\,\phi^m(\xi)+\phi^m(\xi-1)]+\phi(\xi)(1-\phi(\xi))} \qquad \text{(FW)}$$

let $\mu\in\mathbb{R}$, such that $\mu>(2md+1)/c$, $d>0$ and

$$H[\phi](\xi):=\mu\,\phi(\xi)+\frac{d}{c}[\phi^m(\xi+1)-\phi^m(\xi)+\phi^m(\xi-1)]+\frac{1}{c}\phi(\xi)(1-\phi(\xi)), \quad \xi\in\mathbb{R}$$

Function space $S=\{\phi\,|\,\phi(-\infty)=0,\ \phi(\infty)=1,\ \phi'\geq 0\}$

Lemma 1: (H is order-preserving and nondecreasing)
let $\phi\in S$, $\psi:\mathbb{R}\to\mathbb{R}$ *such that* $\phi\leq\psi\leq 1$, *then* $H[\phi](t)\leq H[\psi](t)\ \forall\, t\in\mathbb{R}$;
Moreover $H[\phi]$ *is nondecreasing.*

3. Traveling wavefronts

Lemma 2: ϕ *satisfies* (FW) *if and only if it satisfies*

$$\boxed{\phi(\xi) = \int_{-\infty}^{\xi} e^{\mu s} H[\phi](s)\, ds} \qquad \text{(IFW)}$$

Step 3: upper and lower solutions

Def.: $\phi : \mathbb{R} \to [0,1]$ a.e. differentiable is an uppersolution of (FW) if

$$c\,\phi'(\xi) \geq d\left[\phi^m(\xi+1) - 2\phi^m(\xi) + \phi^m(\xi-1)\right] + \phi(\xi)(1-\phi(\xi))$$

If instead of $\geq$ one has $\leq$ then ϕ is called a lower solution.

3. Traveling wavefronts

Proposition:

(a) *let* $m \geq 2$ *and* $d \leq (4\sinh^2(m/2c))^{-1}$ *then* $\phi^+(\xi) := min\{e^{\xi/c}, 1\}$

is an upper solution of (FW).

(b) *let* $m>1$ *, then for any* ε*,* $0 < \varepsilon < \min\{m-1, 1\}$ *and* M *sufficiently large,*

$$\phi^-(\xi) := max\{0, (1 - Me^{\epsilon\xi/c})e^{\xi/c}\}$$

is a lower solution of (FW).

Note that:

$$0 \leq \phi^-(\xi) \leq \phi^+(\xi) \leq 1 \ \ \forall \xi \in \mathbb{R}, \quad \phi^- \neq 0, \quad \phi^+(-\infty) = 0, \quad \phi^+(+\infty) = 1, \quad \phi^{+\prime}(\xi) \geq 0$$

3. Traveling wavefronts

Step 4: iterative scheme $\phi_1(\xi) := e^{-\mu\xi} \int_{-\infty}^{\xi} e^{\mu s} H[\phi^+](s)\, ds\,, \quad \xi \in \mathbb{R}$

$$\phi_{k+1}(\xi) := e^{-\mu\xi} \int_{-\infty}^{\xi} e^{\mu s} H[\phi_k](s)\, ds\,, \quad \xi \in \mathbb{R}\,, \ k \in \mathbb{N}$$

Proposition 2:

(a) $\phi_1'(\xi) \geq 0\,, \quad \phi^-(\xi) \leq \phi_1(\xi) \leq \phi^+(\xi)\,, \quad \xi \forall \in \mathbb{R}$

(b) $\phi_{k+1}'(\xi) \geq 0\,, \quad \phi^-(\xi) \leq \phi_{k+1}(\xi) \leq \phi_k(\xi) \leq \phi^+(\xi)\,, \quad \xi \forall \in \mathbb{R}$

(c) $\lim_{k\to\infty} \phi_k(\xi) = \phi(\xi)$ *(limit exists),* $\phi^- \leq \phi \leq \phi^+\,,$ ϕ *is non decreasing,*

$$\phi(-\infty) = 0\,, \quad \phi(\infty) = 1$$

3. Traveling wavefronts

Theorem (Fu, Guo, Shieh, 2002)

For DPME $\dot{u}_j = d(u_{j+1}^m - 2u_j^m + u_{j-1}^m) + u_j(1-u_j)\,, \quad j \in \mathbb{Z}$

(a) *for each c>0, m≥2 and* $d \le (4\sinh^2(m/2c))^{-1}$, *there exists a wavefront traveling at speed c*

(b) *for each c>0, 2>m>1 and* $d < sup_{r>0}(rc-1)(4\sinh^2(mr/2))^{-1}$ *, there exists a wavefront traveling at speed c.*

Chen and Guo (2002) Asymptotic stability of traveling wavefronts.
____________ (2003) general monostable reaction terms.
Chen, Fu and Guo (2006) uniqueness of traveling fronts for given *c*.

4. Open questions

• Applicability of MIM is limited to $m \geq 2$ and Fisher-like reaction terms .
Can we design a homotopy argument such that it is applicable to more general terms?

• Can MIM be applied to table A semidiscrete models? How does the dynamics of these models compare against the dynamics of the "classical semidiscretization"? Should we instead work with DSPE or DM-PE? (MIM)

• Start systematic study from DPME (no reaction terms). Interesting points:
Single-pulse response, waiting times, confinement. (work in progress)

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