Perfect state transfer, integral circulants and join of graphs

Ricardo Javier Angeles-Canul* Rachael M. Norton† Michael C. Opperman‡
Christopher C. Paribello§ Matthew C. Russell¶ Christino Tamon‖

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Abstract

We propose new families of graphs which exhibit quantum perfect state transfer. Our constructions are based on the join operator on graphs, its circulant generalizations, and the Cartesian product of graphs. We build upon the results of Bašić *et al.* [5, 4] and construct new integral circulants and regular graphs with perfect state transfer. More specifically, we show that the integral circulant $\mathrm{ICG}_n(\{2, n/2^b\} \cup Q)$ has perfect state transfer, where $b \in \{1, 2\}$, n is a multiple of 16 and Q is a subset of the odd divisors of n. Using the standard join of graphs, we also show a family of double-cone graphs which are non-periodic but exhibit perfect state transfer. This class of graphs is constructed by simply taking the join of the empty two-vertex graph with a specific class of regular graphs. This answers a question posed by Godsil [9].

Keywords: Perfect state transfer, quantum networks, graph join, integral circulants.

1 Introduction

In quantum information systems, the transfer of quantum states from one location to another is an important feature. The problem is to find an arrangement of n interacting qubits in a network which allows perfect transfer of any quantum state over various distances. The network is typically described by a graph where the vertices represent the location of the qubits and the edges represent the pairwise coupling of the qubits. The graph has two special vertices labeled a and b which represent the input (source) and output (target) qubits, respectively. In most cases of interest, it is required that perfect state transfer be achieved without dynamic control over the interactions between the qubits. These are the so-called permanently coupled (unmodulated) spin networks.

*Centro de Investigación en Matemáticas, Universidad Autónoma del Estado de Hidalgo, Pachuca, Hidalgo, Mexico. email: richywhitedragon@gmail.com

†Dept. Mathematics, Bowdoin College, Brunswick, Maine, U.S.A. email: rnorton@bowdoin.edu

‡Dept. Mathematics and Computer Science, Clarkson University, Potsdam, New York, U.S.A. email: oppermmc@clarkson.edu

§Dept. Mathematics and Computer Science, Clarkson University, Potsdam, New York, U.S.A. email: paribecc@clarkson.edu

¶Dept. Mathematics, Taylor University, Upland, Indiana, U.S.A. email: matthew_russell@taylor.edu

‖Dept. Mathematics and Computer Science, Clarkson University, Potsdam, New York, U.S.A. email: tino@clarkson.edu. Corresponding author.

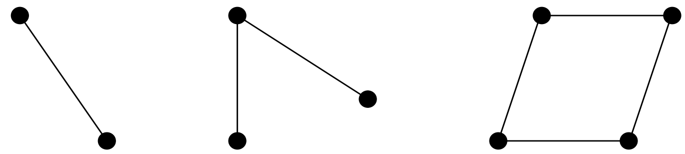

Figure 1: Small perfect state transfer graphs. From left to right: (a) K_2; (b) P_3; (c) C_4.

We may view the perfect state transfer problem in the context of quantum walks on graphs [8, 11]. Here the initial state of the quantum system is described by the unit vector on vertex a. To achieve perfect transfer to vertex b at time t, the quantum walk amplitude of the system at time t on vertex b must be of unit magnitude. That is, to obtain perfect transfer or unit fidelity, we require

$$|\langle b|e^{-itA_G}|a\rangle| = 1, \tag{1}$$

where A_G is the adjacency matrix of the underlying graph G. The main goal here is to characterize graph structures which allow such perfect state transfer.

Christandl *et al.* [6] showed that the Cartesian products of paths of length two or three possess perfect state transfer between antipodal vertices – vertices at maximum distance from each other. They also noted that paths of length four or larger do not possess perfect state transfer if the edges are weighted equally. But, Christandl *et al.* [7] showed that a layered path-like graph of diameter n has perfect state transfer, for any n. More recently, Bernasconi *et al.* [3] gave a complete characterization for graphs from the hypercube family. They proved that perfect state transfer on the generalized n-cube is possible at time $t = \pi/2$ between "antipodal" vertices. Here, "antipodal" depends on the particular sequence that defines the generalized n-cube.

Saxena *et al.* [14], Tsomokos *et al.* [16], and Bašić *et al.* [5] studied perfect state transfer in integral circulant graphs. Tsomokos *et al.* [16] showed perfect state transfer in the class of cross polytope or cocktail party graphs. Bašić *et al.* [5] completely characterized perfect state transfer on unitary Cayley graphs which are equivalent to the integral circulants $\mathrm{ICG}_n(\{1\})$ whose arc lengths must be relatively prime to n. They proved that K_2 and C_4 are the only unitary Cayley graphs with perfect state transfer. Recently, Bašić and Petković [4] proved that the integral circulants $\mathrm{ICG}_n(\{1, n/4\})$ and $\mathrm{ICG}_n(\{1, n/2\})$, for n divisible by 8, have perfect state transfer. In the latter family of integral circulants, we have an example of graphs with perfect state transfer between non-antipodal vertices – vertices which are not at maximum distance from each other. This answers a question posed by Godsil [9]. In this paper, we construct new integral circulants with perfect state transfer by utilizing the join and Cartesian product of graphs.

First, we generalize the graph join to an operation we call the *circulant join* $G +_{\mathrm{C}} G$ between a circulant graph G and a Boolean circulant matrix C. This operation allows us to interpolate between the standard join $G + G$ and the bunkbed (hypercube) operator $K_2 \oplus G$ and, under certain conditions, will produce new circulant graphs. We recover the Cartesian product $K_2 \oplus G$ by taking $C = I$, and the standard join $G + G$ (where all edges between vertices from the distinct copies of G are present) by taking $C = J$ (the all-one matrix). If G has perfect state transfer at time $t^\star$, then so does $G +_{\mathrm{C}} G$ at time

$t^\star$ provided $\cos(t^\star\sqrt{C^T C}) = \pm I$. Moreover, $G +_{\rm C} G$ is a circulant graph whenever C is a palindrome circulant; that is, the sequence which defines C is a palindrome. This allows us to construct new families of circulants with perfect state transfer, namely, $\mathrm{ICG}_n(\{2, n/2^b\} \cup Q)$, $b \in \{1, 2\}$, where n is divisible by 16 and Q is a subset of the odd divisors of n. This expands the class of known integral circulants which exhibit perfect state transfer.

Next, we study graph operators that preserve perfect state transfer. A known example is the Cartesian product of graphs as observed by Christandl *et al.* [6] in the n-fold Cartesian products of paths of length two and three, namely, $K_2^{\oplus n}$ and $P_3^{\oplus n}$. First, we note that this observation can be generalized to Cartesian products of different perfect state transfer graphs $\bigoplus_k G_k$ assuming these graphs have the same perfect state transfer times. We also prove closure properties of the m-fold self-join $\sum_{k=1}^{m} G$ of a graph G with itself. In part, this is one generalization of the standard join $G+G$ we consider in this work. Using these results, we construct new graphs with perfect state transfer, such as $G^{\oplus m}$ and G^{+m}, where G is one of $\mathrm{ICG}_n(\{1, n/2^b\})$ or $\mathrm{ICG}_{2n}(\{2, n/2^{b-1}\} \cup Q)$, where n is a multiple of 8, $b \in \{1, 2\}$, and Q is any subset of the odd divisors of n. These new graphs, however, are not necessarily circulants.

Finally, we consider the join of two arbitrary regular graphs. Bose *et al.* [2] studied perfect state transfer on the complete graphs K_m in the so-called XYZ interaction model. Here, the quantum walk evolves according to the Laplacian of the underlying graph instead of the adjacency matrix (the XY model). They show that, although K_m does not have perfect state transfer, the double-cone $\overline{K}_2 + K_{m-2}$ does. The latter graph is obtained from K_m by removing an edge, say (a, b), and perfect state transfer occurs between a and b. We study a generalization of their construction by considering the join $G + H$ of two arbitrary regular graphs. We show that the existence of perfect state transfer on $G+H$ can be reduced to its existence in G along with some additional conditions on the sizes and regularities of G and H. These conditions are independent of the internal structures of the graphs.

Using this result, we construct a family of double-cone *non*-periodic graphs with perfect state transfer which answers a question posed by Godsil [9]. We also study the double-cone graphs $\overline{K}_2 + G$, $K_2 + G$, for any n-vertex k-regular graph G. We derive sufficient conditions on n and k which allow perfect state transfer between the two special vertices. This complements results found in [2] for the Laplacian model. Our constructions involving $K_2 + G$ also showed that perfect state transfer between non-antipodal vertices is possible. As in the case of $\mathrm{ICG}_n(\{1, n/2\})$, this answers Godsil's other question [9].

Our work heavily exploits the spectral properties of the underlying graphs and their matrices. It is also based on the number-theoretic tools used to characterize integral circulants. A more complete treatment of this beautiful connection between circulants, number theory and graph theory can be found in earlier works by So [13], Saxena *et al.* [14], and Bašić *et al.* [5, 4].

2 Preliminaries

For a logical statement $\mathcal{S}$, the Iversonian $[\![\mathcal{S}]\!]$ is 1 if $\mathcal{S}$ is true and 0 otherwise. Let $\mathbb{Z}_n$ denote the additive group of integers $\{0, \ldots, n-1\}$ modulo n. We use I and J to denote the identity and all-one matrices, respectively; we use X to denote the Pauli-σ_X matrix.

The graphs $G = (V, E)$ we study are finite, simple, undirected, and connected. The

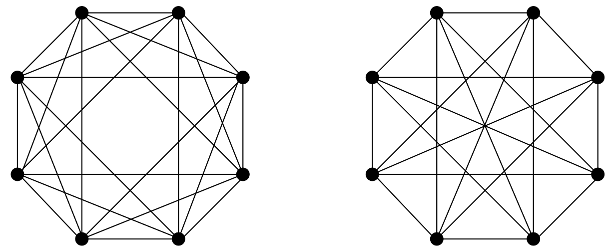

Figure 2: Integral circulants with perfect state transfer. From left to right: (a) $\mathrm{ICG}_8(\{1,2\})$; (b) $\mathrm{ICG}_8(\{1,4\})$. Perfect state transfer occurs from x to $x+4$ at time $\pi/2$ in both graphs (see [4]).

adjacency matrix A_G of a graph G is defined as $A_G[u,v] = [\![(u,v) \in E]\!]$. A graph is called *integral* if its adjacency matrix has only integer eigenvalues. A graph G is *circulant* if its adjacency matrix A_G is circulant. A circulant matrix A is completely specified by its first row, say $[a_0, a_1, \ldots, a_{n-1}]$, and is defined as $A[j,k] = a_{k-j \pmod n}$, where $j, k \in \mathbb{Z}_n$:

$$A = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & \ldots & a_{n-2} \\ \vdots & \vdots & \ldots & \vdots \\ a_1 & a_2 & \ldots & a_0 \end{bmatrix} \tag{2}$$

Note that $a_0 = 0$, since our graphs are simple, and $a_j = a_{n-j}$, since our graphs are undirected. The best known families of circulant graphs include the complete graphs K_n and cycles C_n.

Alternatively, a circulant graph $G = (V,E)$ can be specified by a subset $S \subseteq \mathbb{Z}_n$, where $(j,k) \in E$ if $k - j \in S$. Thus, S defines the set of *edge distances* between adjacent vertices. In this case, we write $G = G(n,S)$. We will assume that S is closed under taking inverses, namely, if $d \in S$, then $-d \in S$. For a divisor d of n, let $G_n(d) = \{k : \gcd(n,k) = d, 1 \leq k < n\}$. It was proved by So [13] that a circulant $G(n,S)$ is integral if and only if $S = \bigcup_{d \in D} G_n(d)$, for some subset D of D_n, where $D_n = \{d : d|n, 1 \leq d < n\}$ is the set of divisors of n. That is, a circulant is integral if its edge distances are elements of $G_n(d)$, $d \in D$, for some subset $D \subseteq D_n$. We denote this family of integral circulants as $\mathrm{ICG}_n(D)$ (following the notation used in [5]).

All circulant graphs G are diagonalizable by the Fourier matrix F whose columns $|F_k\rangle$ are defined as $\langle j|F_k\rangle = \omega_n^{jk}/\sqrt{n}$, where $\omega_n = \exp(2\pi i/n)$. In fact, we have $FAF^\dagger = \sqrt{n} \cdot diag(FA_0)$, for any circulant A, where $A_0 = A|0\rangle$ is the first column of A. This shows that the eigenvalues of A are given by

$$\lambda_j = \sum_{k=0}^{n-1} a_{n-k}\ \omega_n^{jk}. \tag{3}$$

The Cartesian product $G \oplus H$ of graphs G and H is a graph whose adjacency matrix is $I \otimes A_H + A_G \otimes I$. The *join* $G + H$ of graphs G and H is defined as $\overline{G+H} = \overline{G} \cup \overline{H}$; that is, we connect all vertices of G with all vertices of H. The adjacency matrix of $G+H$ is given by $\begin{bmatrix} A_G & J \\ J & A_H \end{bmatrix}$, with the appropriate dimensions on the two all-one J matrices. For more background on algebraic graph theory, we refer the reader to the monographs by Biggs and by Godsil and Royle [1, 10] as well as to the survey article by Schwenk and Wilson [15].

For a graph $G = (V, E)$, let $|\psi(t)\rangle \in \mathbb{C}^{|V|}$ be a time-dependent amplitude vector over V. Then the continuous-time quantum walk on G is defined using Schrödinger's equation as

$$|\psi(t)\rangle = e^{-itA_G}|\psi(0)\rangle, \tag{4}$$

where $|\psi(0)\rangle$ is the initial amplitude vector (see [8]). Further background on quantum walks on graphs can be found in the survey by Kendon [11]. We say G has *perfect state transfer* from vertex a to vertex b at time $t^\star$ if

$$|\langle b|e^{-it^\star A_G}|a\rangle| = 1, \tag{5}$$

where $|a\rangle$, $|b\rangle$ denote the unit vectors corresponding to the vertices a and b, respectively. The graph G has perfect state transfer if there exist vertices a and b in G and a time $t^\star$ so that (5) is true. Also, we call a graph G *periodic* if for any state $|\psi\rangle$, there is a time $t^\star$ so that $|\langle\psi|e^{-itA_G}|\psi\rangle| = 1$.

3 Circulant Joins

In this section, we describe a new graph operator which preserves perfect state transfer. For a n-vertex graph G and a $n \times n$ Boolean matrix C, define the circulant join $\mathcal{G} = G +_{\text{C}} G$ as a graph whose adjacency matrix is

$$A_{\mathcal{G}} = \begin{bmatrix} A_G & C \\ C^T & A_G \end{bmatrix}. \tag{6}$$

That is, we take two copies of G and connect vertices from the corresponding copies using the matrix C. Here, we do not require that C be the adjacency matrix of a graph. This generalizes the join $G + G = G +_{\text{J}} G$ and the bunkbed $K_2 \oplus G = G +_{\text{I}} G$. For these self-join constructions of a graph G with itself, where there are two copies of G, if u is a vertex of G, then we denote (u, s), $s \in \{0, 1\}$, as the vertex u in the s-th copy of G.

Theorem 1 *Let C be a $n \times n$ circulant matrix. If G is a n-vertex circulant graph with perfect state transfer from a to b at time $t^\star$, then the circulant join $G +_C G$ has perfect state transfer from vertex $(a, 0)$ to vertex (b, s), $s \in \{0, 1\}$, at time $t^\star$ provided that*

$$\left[\cos(t^\star\sqrt{B})\right]^{1-s} \left[\sin(t^\star\sqrt{B})B^{-1/2}C^T\right]^{s} = \pm I \tag{7}$$

where $B = C^T C$, and B^{-1} exists whenever $s = 1$. Moreover, $G +_C G$ is a circulant graph if C is a palindrome circulant matrix, where $c_j = c_{n-1-j}$, for $j = 0, \ldots, n-1$.

Proof Note that the adjacency matrix of $\mathcal{G} = G +_{\text{C}} G$ can be rearranged as

$$\mathcal{C}_A = \left(C \otimes |0\rangle\langle 1| + C^T \otimes |1\rangle\langle 0|\right) + A_G \otimes I_2. \tag{8}$$

It is clear that C and C^T commute since they are both circulants. Next, observe that

$$\left[C \otimes |0\rangle\langle 1| + C^T \otimes |1\rangle\langle 0|\right]^\ell = \begin{cases} B^k \otimes I_2 & \text{if } \ell = 2k \\ B^k C \otimes |0\rangle\langle 1| + B^k C^T \otimes |1\rangle\langle 0| & \text{if } \ell = 2k+1 \end{cases} \tag{9}$$

In the above equation, notice that the even or odd powers vanish depending on s:

$$\langle b|\langle 0| \left[B^k C \otimes |0\rangle\langle 1| + B^k C^T \otimes |1\rangle\langle 0|\right] |a\rangle|0\rangle = 0. \tag{10}$$

and

$$\langle b|\langle 1|B^k \otimes I_2|a\rangle|0\rangle = 0. \tag{11}$$

Thus, perfect state transfer in $\mathcal{G}$ can be reduced to perfect state transfer in G as follows:

$$\begin{aligned} \langle b, s|e^{-it^\star \mathcal{C}_A}|a, 0\rangle &= \langle b|\langle s|e^{-it^\star(C\otimes|0\rangle\langle 1|+C^T\otimes|1\rangle\langle 0|)}e^{-it^\star(A_G\otimes I_2)}|a\rangle|0\rangle & (12) \\ &= \langle b|\langle s|e^{-it^\star(C\otimes|0\rangle\langle 1|+C^T\otimes|1\rangle\langle 0|)}(e^{-it^\star A_G} \otimes I_2)|a\rangle|0\rangle & (13) \\ &= \begin{cases} \langle b|\cos(t^\star\sqrt{B})e^{-it^\star A_G}|a\rangle & \text{if } s = 0 \\ -i\langle b|\sin(t^\star\sqrt{B})B^{-1/2}C^T e^{-it^\star A_G}|a\rangle & \text{if } s = 1 \end{cases} & (14) \end{aligned}$$

This proves the first claim.

To see that $\mathcal{G}$ is a circulant graph if C is a palindrome circulant matrix, we view the adjacency matrix $\mathcal{C}_A$ as an "interweaving" of A_G with C and C^T as follows:

$$\mathcal{C}_A = \begin{bmatrix} a_0 & \langle c_0\rangle & a_1 & \langle c_1\rangle & a_2 & \langle c_2\rangle & \dots & a_{n-1} & \langle c_{n-1}\rangle \\ [c_0] & a_0 & [c_{n-1}] & a_1 & [c_{n-2}] & a_2 & \dots & [c_1] & a_{n-1} \\ a_{n-1} & \langle c_{n-1}\rangle & a_0 & \langle c_0\rangle & a_1 & \langle c_1\rangle & \dots & a_{n-2} & \langle c_{n-2}\rangle \\ [c_1] & a_{n-1} & [c_0] & a_0 & [c_{n-1}] & a_1 & \dots & [c_2] & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_1 & \langle c_1\rangle & a_2 & \langle c_2\rangle & a_3 & \langle c_3\rangle & \dots & a_0 & \langle c_0\rangle \\ [c_{n-1}] & a_1 & [c_{n-2}] & a_2 & [c_{n-3}] & a_3 & \dots & [c_0] & a_0 \end{bmatrix} \tag{15}$$

where $A_G = (a_j)$, $C = (\langle c_j\rangle)$, and $C^T = ([c_j])$. We have distinguished the elements of C and C^T by using $\langle c\rangle$ and $[c]$, respectively. Now, applying $c_j = c_{n-1-j}$, for $j = 0, \dots, n-1$, it is clear that the above is a circulant matrix; hence, $\mathcal{G}$ is a circulant graph. □

As corollaries to the above theorem, we show the conditions for which the hypercube Cartesian product and the join of a perfect state transfer graph with itself preserves the perfect state transfer property.

Corollary 2 *If G is an n-vertex circulant that has perfect state transfer from a to b at time $t^\star \in (2\mathbb{Z}+1)\frac{\pi}{2}$, then the bunkbed $K_2 \oplus G$ has perfect state transfer from $(a, 0)$ to $(b, 1)$, where $(a, 0)$ denotes vertex a in the first copy of G and $(b, 1)$ denotes vertex b in the second copy of G.*

Proof Since $K_2 \oplus G = G +_{\text{I}} G$, the eigenvalues of the connection matrix are $\mu_k = 1$ for all k. By Theorem 1, we require $\sin(t^\star I_n) = \pm I_n$ for perfect state transfer. This is equivalent to $\sin(t^\star) = \pm 1$ which is satisfied whenever $t^\star \in (2\mathbb{Z}+1)\frac{\pi}{2}$. □

Corollary 3 *If G is an n-vertex circulant graph that has perfect state transfer from a to b at time $t^\star$, then so does the circulant graph $G+G$ provided $nt^\star \in 2\pi\mathbb{Z}$.*

Proof Since $G+G = G +_{\mathrm{J}} G$, by Theorem 1 we require $\cos(t^\star \Lambda) = I_n$, where $\Lambda = diag(n, 0, \ldots, 0)$. This is equivalent to requiring $\cos(nt^\star) = 1$ which is satisfied when $nt^\star \in 2\pi\mathbb{Z}$. □

Remark: It can be shown that for $n = 2^u$, where $u \geq 3$, the only Boolean circulant matrices C that yield a circulant graph $G +_{\mathrm{C}} G$, for an n-vertex G, are the trivial matrices, namely, $C \in \{I_n, J_n, O_n\}$, where O_n is the $n \times n$ all-zero matrix. In the next theorem, we show that for n that is a multiple of 8, if n has a non-trivial odd divisor, then there exist integral circulant graphs $\mathrm{ICG}_{2n}(D)$, for $|D| \geq 3$, with perfect state transfer which are obtained from non-trivial circulant joins.

Notation: For an integer m, let $\overline{D}_m = \{d : d|m, 1 \leq d \leq m\}$ be the set of all divisors of m. Also, for an integer k and a set $A \subseteq \mathbb{Z}$, we use kA to denote $\{ka : a \in A\}$.

Theorem 4 *Let $n = 2^u m$, where $u \geq 3$ and $m \geq 3$ is an odd number. Suppose that $G = \mathrm{ICG}_n(D)$, for $D = \{1, n/4\}$ or $D = \{1, n/2\}$. For any subset $Q \subset \overline{D}_m$, there is a Boolean circulant matrix $C \notin \{I_n, J_n, O_n\}$ so that*

$$G +_C G = \mathrm{ICG}_{2n}(2D \cup Q) \tag{16}$$

has perfect state transfer from 0 to $n/2$ in G at time $t^\star = \pi/2$,

Proof For $q \in Q$, let $N(q) = \{r \in \overline{D}_m : r/q \text{ is an odd prime }\}$. Define, for $j = 0, \ldots, n-1$,

$$c_j(q) = [\![2j+1 \equiv 0 \pmod{q} \wedge \forall r \in N(q) : 2j+1 \not\equiv 0 \pmod{r}]\!]. \tag{17}$$

Now let $c_j(Q) = \sum_{q \in Q} c_j(q)$. Note that $c_j(q)$'s are disjoint, since at most one index j will satisfy $c_j(q) = 1$ for $q \in Q$. The Boolean circulant C is defined by the following first row

$$C = [c_0(Q) \ \ldots \ c_{n-1}(Q)]. \tag{18}$$

To see that C is a palindrome, note that $2(n-1-j)+1 \equiv 0 \pmod{q}$ is equivalent to $2j+1 \equiv 0 \pmod{q}$, for any $q \in \overline{D}_m$.

Since the integral circulants $\mathrm{ICG}_n(\{1, n/2^b\})$, $b \in \{1,2\}$, have $t^\star = \pi/2$ as the perfect state transfer time, we will show that $\cos(|\lambda_k|\pi/2) = 1$, for all eigenvalues λ_k of C, $k = 0, \ldots, n-1$. The eigenvalues of C are given by

$$\lambda_k = \sum_{j=0}^{n-1} c_j(Q)\omega_n^{-jk} = \sum_{j=0}^{n-1}\sum_{q\in Q} c_j(q)\omega_n^{-jk} = \sum_{q\in Q}\sum_{j=0}^{n-1} c_j(q)\omega_n^{-jk} \tag{19}$$

$$= \sum_{q\in Q}\left\{\sum_{\ell=0}^{\frac{n}{q}-1} \omega_n^{-(\lfloor q/2\rfloor + \ell q)k} - \sum_{r\in N(q)}\sum_{\ell=0}^{\frac{n}{r}-1} \omega_n^{-(\lfloor r/2\rfloor + \ell r)k}\right\} \tag{20}$$

$$= \sum_{q\in Q}\left\{\frac{n}{q}\omega_n^{-\lfloor q/2\rfloor k}[\![k \equiv 0 \pmod{n/q}]\!] - \sum_{r\in N(q)} \frac{n}{r}\omega_n^{-\lfloor r/2\rfloor k}[\![k \equiv 0 \pmod{n/r}]\!]\right\} \tag{21}$$

Now, we consider the exponent in the term $\omega_n^{-\lfloor q/2 \rfloor k}$, where k satisfies $k \equiv 0 \pmod{n/q}$. Thus, there is an integer κ so that $qk = \kappa n$ or $(2\lfloor q/2 \rfloor + 1)k = \kappa n$. Note that κ depends on k. After rearranging, we get $-\lfloor q/2 \rfloor k = \frac{k}{2} - \kappa \frac{n}{2}$. Thus,

$$\omega_n^{-\lfloor q/2 \rfloor k} = \omega_n^{k/2} (\omega_n^{-n/2})^{\kappa} = \omega_n^{k/2} (-1)^{\kappa}. \tag{22}$$

Thus, we have

$$\lambda_k \in \omega_n^{k/2} (-1)^{\kappa} 2^u \mathbb{Z}. \tag{23}$$

It is clear now that $\cos(|\lambda_k|\pi/2) = \cos(2\pi\mathbb{Z}) = 1$, for all $k = 0, \ldots, n-1$. So, by Theorem 1, $G +_{\mathrm{C}} G$ has perfect state transfer at time $\pi/2$.

Now, we show that if $G = \mathrm{ICG}_n(D)$, then $G +_{\mathrm{C}} G = \mathrm{ICG}_{2n}(2D \cup Q)$, where $2D = \{2d : d \in D\}$. Let B be the circulant adjacency matrix of $G +_{\mathrm{C}} G$ defined by the sequence $[b_0, \ldots, b_{2n-1}]$. From the "interweaving" property of B in (15), we know that for $k \in \{0, \ldots, 2n-1\}$

$$b_k = \begin{cases} a_{k/2} & \text{if } k \text{ is even} \\ c_{\lfloor k/2 \rfloor} & \text{if } k \text{ is odd} \end{cases} \tag{24}$$

We consider two cases based on whether k is even or odd. For k odd, we have

$$\begin{aligned} b_k &= c_{\lfloor k/2 \rfloor} && (25) \\ &= [\![\exists q \in Q : 2\lfloor k/2 \rfloor + 1 \equiv 0 \,(\mathrm{mod}\; q) \wedge \forall r \in N(q) : 2\lfloor k/2 \rfloor + 1 \not\equiv 0 \,(\mathrm{mod}\; r)]\!] && (26) \\ &= [\![\exists q \in Q : k \equiv 0 \,(\mathrm{mod}\; q) \wedge \forall r \in N(q) : k \not\equiv 0 \,(\mathrm{mod}\; r)]\!] && (27) \\ &= [\![\exists q \in Q : \gcd(k, 2n) = q]\!], && (28) \end{aligned}$$

whereas for k even, we have

$$b_k = a_{k/2} = [\![\gcd(k/2, n) \in D]\!] = [\![\gcd(k, 2n) \in 2D]\!]. \tag{29}$$

This proves the claim by taking $D = \{1, n/4\}$ or $D = \{1, n/2\}$. □

Corollary 5 *For $n = 2^u$, for $u \geq 3$, the integral circulant graphs $\mathrm{ICG}_{2n}(\{1, 2, n/2\})$ and $\mathrm{ICG}_{2n}(\{1, 2, n\})$ have perfect state transfer from 0 to n at time $t^\star = \pi/2$.*

Proof Take the self-joins of $\mathrm{ICG}_n(\{1, n/4\})$ and $\mathrm{ICG}_n(\{1, n/2\})$ which have perfect state transfer as shown by Bašić and Petković [4]. □

In the next corollary, we consider the class of circulant permutation matrices C. These matrices are defined by specifying a unit vector (with a one in a single entry and zeros elsewhere) as their first row, and they satisfy $C^T C = I$. We show that if G has perfect state transfer from a to b at time $t^\star$, then so does $G +_{\mathrm{C}} G$ from a in the first copy of G to Cb (image of vertex b under the permutation C) in the second copy of G at time $t^\star$.

Corollary 6 *Let G be a n-vertex graph and let C be a $n \times n$ circulant permutation matrix. If G has perfect state transfer from vertex a to b at time $t^\star$, then $G +_C G$ has perfect state transfer from vertex $(a, 0)$ to $(Cb, 1)$ at time $t^\star$, where Cb is the image of vertex b under the permutation C.*

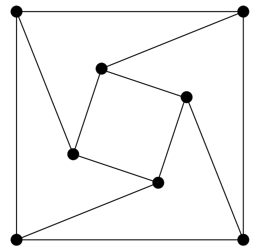

Figure 3: Standard circulant joins on C_4. From left to right: (a) Cartesian bunkbed $C_4 +_{\text{I}} C_4$; (b) Self-join $C_4 +_{\text{J}} C_4$

Proof Let $\mathcal{G} = G +_{\text{C}} G$. Since C is a permutation matrix, we have $C^T C = I$. Applying (14) in the proof of Theorem 1, we get

$$\langle b|\langle 1|e^{-itA_{\mathcal{G}}}|a\rangle|0\rangle = -i\langle b|C^T e^{-itA_G}|a\rangle. \tag{30}$$

which can be rearranged as $\langle Cb|\langle 1|e^{-itA_{\mathcal{G}}}|a\rangle|0\rangle = -i\langle b|e^{-itA_G}|a\rangle$. □

Remark: The above corollary shows that if there is perfect state transfer in G from vertex a to vertex b then there is perfect state transfer from $(a, 0)$ to any vertex $(c, 1)$ in $G +_{\text{C}} G$. This is achieved by choosing a permutation matrix C so that $Cb = c$.

4 Cartesian Products and Self-Joins

We show that a heterogeneous Cartesian product of perfect state transfer graphs has perfect state transfer. This generalizes the observations on the hypercube $K_2^{\oplus n}$ and the Cartesian product $P_3^{\oplus n}$ of paths of length three (see [6]).

Theorem 7 *For $j = 1, \ldots, m$, the graph G_j has perfect state transfer from a_j to b_j at time $t^\star$ if and only if $\mathcal{G} = \bigoplus_{j=1}^{m} G_j$ has perfect state transfer from $(a_1, \ldots, a_m)$ to $(b_1, \ldots, b_m)$ at time $t^\star$.*

Proof We prove the claim for $m = 2$. Then,

$$\begin{aligned}
\langle b_1, b_2|e^{-itA_{G_1 \oplus G_2}}|a_1, a_2\rangle &= \langle b_1, b_2|e^{-it(I \otimes A_{G_2} + A_{G_1} \otimes I)}|a_1, a_2\rangle && (31)\\
&= \langle b_1|\langle b_2|e^{-it(I \otimes A_{G_2})}e^{-it(A_{G_1} \otimes I)}|a_1\rangle|a_2\rangle && (32)\\
&= \langle b_1|\langle b_2|(I \otimes e^{-itA_{G_2}})(e^{-itA_{G_1}} \otimes I)|a_1\rangle|a_2\rangle && (33)\\
&= \langle b_1|e^{-itA_{G_1}}|a_1\rangle\langle b_2|e^{-itA_{G_2}}|a_2\rangle. && (34)
\end{aligned}$$

This shows that $G_1 \oplus G_2$ has perfect state transfer from (a_1, a_2) to (b_1, b_2) at time $t^\star$ if and only if G_1 has perfect state transfer from a_1 to b_1 at time $t^\star$ *and* G_2 has perfect state transfer from a_2 to b_2 at time $t^\star$. The general claim follows easily by induction. □

Corollary 8 *For any m and n so that $n \equiv 0 \pmod 8$, the family of graphs $\bigoplus_{k=1}^{m} G_k$, where $G_k \in \{\mathrm{ICG}_n(\{1, n/2^b\}), \mathrm{ICG}_{2n}(\{2, n/2^{b-1}\} \cup Q)\}$, $b \in \{1,2\}$ and Q is a subset of the set of odd divisors of n, has perfect state transfer from vertex 0 to $n/2$ at time $t^\star = \pi/2$.*

Proof Follows from Theorem 7, the results of Bašić *et al.* [4], and Theorem 1. □

We also show that the m-fold join of a perfect state transfer graph preserves perfect state transfer under certain conditions. Denote G^{+m} as the m-fold self-join $\sum_{j=1}^{m} G$.

Theorem 9 *Let G be an n-vertex regular graph. For $m \geq 1$, the existence of perfect state transfer in G^{+m} between vertices a and b (in the same copy of G) can be reduced to its existence in G as follows:*

$$\langle 0, b|e^{-itA_{G^{+m}}}|0, a\rangle = \langle b|e^{-itA_G}|a\rangle + \left[\frac{(m-1)(e^{itn}-1) + e^{-it(m-1)n} - 1}{mn}\right] \langle 1_n|e^{-itA_G}|a\rangle, \tag{35}$$

where A_G is the adjacency matrix of G and $|1_n\rangle$ is the all-one column vector of length n.

Proof The adjacency matrix of G^{+m} is given by $I_m \otimes A_G + K_m \otimes J_n$, where K_m is the adjacency matrix of the complete graph on m vertices. First note that $J_n^\ell = n^{\ell-1} J_n$, if $\ell \geq 1$, and $J_n^0 = I_n$. Also, notice that A_G commutes with J_n since G is a regular graph. Moreover, $K_m = J_m - I_m$. Thus, using the binomial theorem, we get

$$K_m^\ell = \frac{1}{m}\left\{(-1)^\ell (mI_m - J_m) + (m-1)^\ell J_m\right\}. \tag{36}$$

Therefore,

$$\begin{aligned}
e^{-it(K_m \otimes J_n)} &= \sum_{\ell=0}^{\infty} \frac{(-it)^\ell}{\ell!} K_m^\ell \otimes J_n^\ell && (37)\\
&= I_m \otimes I_n + \sum_{\ell=1}^{\infty} \frac{(-it)^\ell}{\ell!} K_m^\ell \otimes J_n^\ell && (38)\\
&= I_m \otimes I_n + \frac{1}{n}\sum_{\ell=1}^{\infty} \frac{(-it)^\ell}{\ell!} K_m^\ell \otimes (n^\ell J_n) && (39)\\
&= I_m \otimes I_n + \frac{(e^{itn}-1)}{mn}(mI_m - J_m) \otimes J_n + \frac{(e^{-it(m-1)n}-1)}{mn} J_m \otimes J_n && (40)
\end{aligned}$$

We can now analyze the quantum walk amplitude from vertex a to b (in the same copy of G). We get

$$\begin{aligned}
\langle 0, b|e^{-itA_{G^{+m}}}|0, a\rangle &= \langle 0|\langle b|e^{-it(K_m \otimes J_n)} e^{-it(I_m \otimes A_G)}|0\rangle|a\rangle && (41)\\
&= \langle 0|\langle b|e^{-it(K_m \otimes J_n)} (I_m \otimes e^{-itA_G})|0\rangle|a\rangle && (42)\\
&= \langle 0|\langle b|e^{-it(K_m \otimes J_n)} (|0\rangle \otimes e^{-itA_G}|a\rangle) && (43)
\end{aligned}$$

Expanding the second term using (40) and multiplying the two terms on the left, we get

$$\left(\langle 0|\langle b| + \frac{(e^{itn}-1)}{mn}(m\langle 0| - \langle 1_m|)\otimes\langle 1_n| + \frac{(e^{-it(m-1)n}-1)}{mn}\langle 1_m|\otimes\langle 1_n|\right)(|0\rangle\otimes e^{-itA_G}|a\rangle). \tag{44}$$

Finally, combining this with the last term, we arrive at

$$\langle b|e^{-itA_G}|a\rangle + \frac{1}{mn}\left[(e^{itn}-1)(m-1) + e^{-it(m-1)n} - 1\right]\langle 1_n|e^{-itA_G}|a\rangle, \tag{45}$$

which proves the claim. □

Corollary 10 *For any $m \geq 1$ and $n \equiv 0 \pmod 8$, the family of graphs G^{+m}, where $G \in \{\mathrm{ICG}_n(\{1, n/2^b\}), \mathrm{ICG}_{2n}(\{2, n/2^{b-1}\} \cup Q)\}$, with $n \equiv 0 \pmod 8$, $b \in \{1, 2\}$, and a subset Q of the odd divisors of n, has perfect state transfer between vertices 0 and $n/2$ (in the same copy of G).*

Proof By Theorem 9, to achieve perfect state transfer in G^{+m}, it suffices to have perfect state transfer in G at time $t^\star$ and have $e^{it^\star n} = 1$. By the results of Bašić *et al.* [4] and by Theorem 4, the integral circulant graphs stated in the claim have perfect state transfer from vertex 0 to vertex $n/2$ at time $t^\star = \pi/2$. Therefore, it suffices to have $n \in 4\mathbb{Z}$. Since $n \equiv 0 \pmod 8$, this holds for any m. □

Corollary 11 *For any $m \geq 1$ and $n \geq 2$, the family of graphs Q_n^{+m}, where Q_n is the binary n-dimensional hypercube, has perfect state transfer between antipodal vertices in the same copy of Q_n.*

Proof Bernasconi *et al.* [3] proved that Q_n has perfect state transfer between its antipodal vertices at time $t^\star = \pi/2$. Note that the number of vertices of Q_n is $N = 2^n$. Using Theorem 9, it suffices to set $e^{it^\star N} = 1$ or $N \equiv 0 \pmod 4$. This is always true since $N = 2^n$, with $n \geq 2$. □

5 Join of Regular Graphs

We show that the existence of perfect state transfer in a join of two arbitrary regular graphs can be reduced to perfect state transfer in one of the graphs along with certain additional constraints on the sizes and degrees of the graphs. These conditions are independent of the internal structures of the graphs.

Theorem 12 *Let G be an m-vertex k_G-regular graph and let H be an n-vertex k_H-regular graph. Suppose that a and b are two vertices in G. Then,*

$$\langle b|e^{-itA_{G+H}}|a\rangle = \langle b|e^{-itA_G}|a\rangle + \frac{e^{-itk_G}}{m}\left\{e^{it\delta/2}\left[\cos\left(\frac{\Delta t}{2}\right) - i\left(\frac{\delta}{\Delta}\right)\sin\left(\frac{\Delta t}{2}\right)\right] - 1\right\} \tag{46}$$

where $\delta = k_G - k_H$ and $\Delta = \sqrt{\delta^2 + 4mn}$.

Proof Let a, b be two vertices of G. Then,

$$\langle b|e^{-itA_G}|a\rangle = \langle b| \left\{ \sum_{k=0}^{m-1} |u_k\rangle\langle u_k|e^{-it\lambda_k} \right\} |a\rangle \tag{47}$$

where λ_k and $|u_k\rangle$ are the eigenvalues and eigenvectors of A_G, for $k = 0, \ldots, m-1$. We assume $|u_0\rangle$ is the all-one eigenvector (that is orthogonal to the other eigenvectors) with eigenvalue $\lambda_0 = k_G$. By the same token, let κ_ℓ and $|v_\ell\rangle$ be the eigenvalues and eigenvectors of A_H, for $\ell = 0, \ldots, n-1$. Also, $|v_0\rangle$ is the all-one eigenvector (with eigenvalue $\kappa_0 = k_H$) which is orthogonal to the other eigenvectors $|v_\ell\rangle$, $\ell \neq 0$.

Let $\mathcal{G} = G + H$. Note that the adjacency matrix of $\mathcal{G}$ is

$$A_{\mathcal{G}} = \begin{bmatrix} A_G & J_{m\times n} \\ J_{n\times m} & A_H \end{bmatrix}. \tag{48}$$

Let $\delta = k_G - k_H$. The eigenvalues and eigenvectors of $A_{\mathcal{G}}$ are given by the three sets:

- For $k = 1, \ldots, m-1$, let $|u_k, 0_n\rangle$ be a column vector formed by concatenating the column vector $|u_k\rangle$ with the zero vector of length n. Then, $|u_k, 0_n\rangle$ is an eigenvector with eigenvalue λ_k.
- For $\ell = 1, \ldots, n-1$, let $|0_m, v_\ell\rangle$ be a column vector formed by concatenating the zero vector of length m with the column vector $|v_\ell\rangle$. Then, $|0_m, v_\ell\rangle$ is an eigenvector with eigenvalue κ_ℓ.
- Let $|\pm\rangle = \frac{1}{\sqrt{L_\pm}}|\alpha_\pm, 1_n\rangle$ be a column vector formed by concatenating the vector $\alpha_\pm|1_m\rangle$ with the vector $|1_n\rangle$, where $|1_m\rangle$, $|1_n\rangle$ denote the all-one vectors of length m, n, respectively. Then, $|\pm\rangle$ is an eigenvector with eigenvalue $\lambda_\pm = k_H + m\alpha_\pm$. Here,

$$\alpha_\pm = \frac{1}{2m}(\delta \pm \Delta), \quad L_\pm = m(\alpha_\pm)^2 + n. \tag{49}$$

In what follows, we will abuse notation by using $|a\rangle$, $|b\rangle$ for both G and $G + H$; their dimensions differ in both cases, although it will be clear from context which version is used. The quantum wave amplitude from a to b is given by

$$\begin{aligned}
\langle b|e^{-itA_{\mathcal{G}}}|a\rangle &= \langle b|e^{-itA_{\mathcal{G}}} \left\{ \sum_{k=1}^{m-1} \langle u_k, 0_n|a\rangle |u_k, 0_n\rangle + \sum_{\pm} \frac{\alpha_\pm}{\sqrt{L_\pm}}|\pm\rangle \right\} && (50) \\
&= \langle b| \left\{ \sum_{k=1}^{m-1} \langle u_k|a\rangle e^{-it\lambda_k} |u_k, 0_n\rangle + \sum_{\pm} \frac{\alpha_\pm}{\sqrt{L_\pm}} e^{-it\lambda_\pm}|\pm\rangle \right\} && (51) \\
&= \sum_{k=1}^{m-1} \langle b|u_k\rangle\langle u_k|a\rangle e^{-it\lambda_k} + \sum_{\pm} \frac{\alpha_\pm^2}{L_\pm} e^{-it\lambda_\pm} && (52) \\
&= \langle b| \left\{ \sum_{k=0}^{m-1} |u_k\rangle\langle u_k|e^{-it\lambda_k} \right\} |a\rangle - \frac{e^{-itk_G}}{m} + \sum_{\pm} \frac{\alpha_\pm^2}{L_\pm} e^{-it\lambda_\pm} && (53) \\
&= \langle b|e^{-itA_G}|a\rangle + \sum_{\pm} \frac{\alpha_\pm^2}{L_\pm} e^{-it\lambda_\pm} - \frac{e^{-itk_G}}{m}. && (54)
\end{aligned}$$

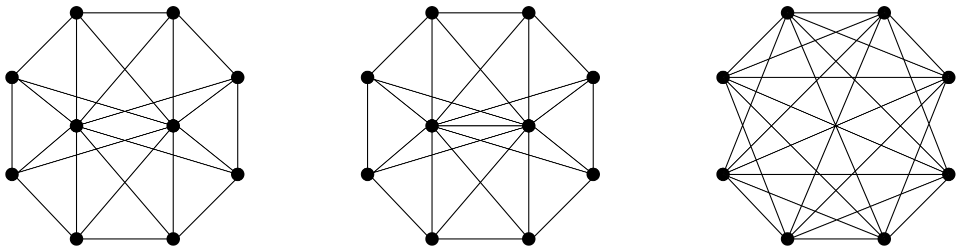

Figure 4: Double Cones. From left to right: (a) $\overline{K}_2 + C_8$; (b) $K_2 + C_8$; (c) Cocktail Party (hyperoctahedral).

We analyze the second term next. Note that we have the following identities:

$$\begin{aligned} \alpha_+\alpha_- &= -(n/m) & (55)\\ \alpha_+ + \alpha_- &= \delta/m & (56)\\ L_+L_- &= (n/m)\Delta^2 & (57)\\ L_+ + L_- &= \Delta^2/m & (58)\\ (\alpha_\pm)^2 L_\mp &= (n/m)L_\pm & (59)\\ \lambda_\pm &= (\hat{\delta} \pm \Delta)/2 & (60) \end{aligned}$$

where $\hat{\delta} = k_G + k_H$. Therefore, the summand in (54) is given by

$$\sum_{\pm} \frac{\alpha_\pm^2}{L_\pm} e^{-it\lambda_\pm} = \frac{1}{m} e^{-it\hat{\delta}/2} \left[\cos\left(\frac{\Delta t}{2}\right) - i\left(\frac{\delta}{\Delta}\right) \sin\left(\frac{\Delta t}{2}\right) \right]. \tag{61}$$

This yields

$$\langle b|e^{-itA_{\mathcal{G}}}|a\rangle = \langle b|e^{-itA_G}|a\rangle + \frac{e^{-itk_G}}{m} \left\{ e^{it\delta/2} \left[\cos\left(\frac{\Delta t}{2}\right) - i\left(\frac{\delta}{\Delta}\right) \sin\left(\frac{\Delta t}{2}\right) \right] - 1 \right\} \tag{62}$$

which proves the claim. □

In what follows, we describe several applications of Theorem 12 to the double-cones $\overline{K}_2 + G$ and $K_2 + G$ and also to the construction of a family of non-periodic graphs with perfect state transfer. The existence of the latter family of graphs was one of the main questions posed by Godsil [9]. For a prime p, we denote $S_p(n)$ to be the largest non-negative integer j so that $p^j|n$.

Corollary 13 *For any k-regular graph G on n vertices, $\overline{K}_2 + G$ has perfect state transfer between the two non-adjacent vertices of $\overline{K}_2$ if $\Delta = \sqrt{k^2 + 8n}$ is an integer and $k, \Delta \equiv 0 \pmod 4$ with $S_2(k) \neq S_2(\Delta)$.*

Proof Let $\mathcal{G} = \overline{K}_2 + G$. By Theorem 12, since there is no transfer between the two vertices of $\overline{K}_2$, we have

$$\langle b|e^{-itA_{\mathcal{G}}}|a\rangle = \frac{1}{2} \left\{ e^{-itk/2} \left[\cos\left(\frac{\Delta t}{2}\right) + i\left(\frac{k}{\Delta}\right) \sin\left(\frac{\Delta t}{2}\right) \right] - 1 \right\}. \tag{63}$$

To achieve unit magnitude, it is necessary and sufficient to require $\cos(kt/2)\cos(\Delta t/2) = -1$. Note that setting $k = 0$ (*i.e.* G is the empty graph on n vertices) will result in this condition being satisfied for $t = 2\pi/\Delta$. This is, in a sense, a generalization of P_3, which has perfect state transfer [6].

Otherwise, assume $k = 2^{k_0}k_1$, where k_1 is odd; and $\Delta = 2^{d_0}d_1$, where d_1 is odd. Since there is no transfer in $\overline{K}_2$, we may choose $t \in \mathbb{Q}\pi$ and require that Δ be an integer. It is clear that $kt/2$ and $\Delta t/2$ must have opposite parities as multiples of π. This implies $k_0 \neq d_0$. If $k_0 > d_0$, we have

$$n = \frac{1}{8}(\Delta^2 - k^2) = \frac{1}{8}(4^{d_0}d_1^2 - 4^{k_0}k_1^2) = \frac{4^{d_0}}{8}(d_1^2 - 4^{k_0-d_0}k_1^2). \tag{64}$$

Since $(d_1^2 - 4^{k_0-d_0}k_1^2)$ is odd and n is an integer, 8 divides 4^{d_0} which implies $d_0 \geq 2$ and $k_0 > 2$. A similar argument when $k_0 < d_0$ shows that $d_0 > k_0 \geq 2$. Thus, both k and Δ are multiples of 4. □

Remark: Using $k \equiv 0 \pmod 4, n = k+2, \Delta = k+4$ satisfy the conditions of Corollary 13, and thus the graph $\mathcal{G} = \overline{K}_2 + G$ has perfect state transfer. In this case, $\mathcal{G}$ can be represented by a type of circulant graph called a *hyperoctahedral*, or *cocktail-party*, graph [1] (see Figure 4). These graphs are formed by removing $n/2+1$ disjoint edges from K_{n+2}. This class of graphs, which is also called the class of cross polytope graphs, was also studied by Tsomokos *et al.* [16].

Next, we answer a question of Godsil [9] (Section 10, question (b)) by constructing an infinite family of non-periodic graphs with the perfect state transfer property.

Corollary 14 *For $\ell \geq 2$, the family of double-cone graphs $\overline{K}_2 + (C_{2(2\ell-1)} \oplus C_{2\ell+1})$ is non-periodic and has perfect state transfer.*

Proof Let $G = C_{2(2\ell-1)} \oplus C_{2\ell+1}$, for $\ell \geq 2$. Note that G is a k-regular graph with $k = 4$ and $n = 2(4\ell^2 - 1)$ vertices. Using the notation of Theorem 12, we have $\Delta = \sqrt{k^2 + 8n} = 8\ell$. The eigenvalues of G are given by the sum of the eigenvalues of the two cycles:

$$\lambda(G) = \lambda(C_{2(2\ell-1)}) + \lambda(C_{2\ell+1}). \tag{65}$$

Recall that the eigenvalues of an n-cycle are given by $2\cos(2\pi k/n)$, for $k = 0, \ldots, n-1$. So, each cycle has 2 (its degree) as its largest eigenvalue. Thus, the sums of the cycle eigenvalues contain both integers and irrational numbers. For $n = 5$ and $n \geq 7$, at least some of these values are irrational. This is because the only rational values of $\cos((a/b)\pi)$, for $a, b \in \mathbb{Z}$, are $\{0, \pm 1/2, \pm 1\}$ (see Corollary 3.12 in Niven [12]). Note that $2(2\ell - 1) \geq 5$ and $2\ell + 1 \geq 5$ hold for $\ell \geq 2$, and that both expressions cannot equal 6.

The eigenvalues of $\mathcal{G} = \overline{K}_2 + G$ will then be all of the eigenvalues of G (except for 4), 0, and $\lambda_{\pm} = \frac{1}{2}(4 \pm 8\ell) = 2 \pm 4\ell$ by (60). This means that $\mathcal{G}$ has a mixture of integral and irrational eigenvalues. By Lemma 4.1 in [9], the graph $\mathcal{G}$ is non-periodic. By Corollary 13, since $\Delta = 8\ell$ is an integer and $S_2(k) \neq S_2(\Delta)$, we know that $\mathcal{G}$ has perfect state transfer. This proves the claim. □

Remark: Taking $\ell = 2$ in Corollary 14, we get $\mathcal{G} = \overline{K}_2 + C_5 \oplus C_6$. Again by Lemma 4.1 in [9], $\mathcal{G}$ is non-periodic since its eigenvalues contain both integers and irrational numbers. By Corollary 14, we know it has perfect state transfer although it violates the *eigenvalue ratio* condition $(\lambda_k - \lambda_\ell)/(\lambda_r - \lambda_s) \in \mathbb{Q}$, for $\lambda_r \neq \lambda_s$. This is in contrast to Theorem 2.1 in [9] and to the mirror-symmetric networks in Section III from [7]. Our double-cone construction is mirror-symmetric with respect to the two vertices of $\overline{K}_2$.

Corollary 15 *For any n-vertex k-regular graph G, let $\tilde{k} = k-1$. Then, $K_2 + G$ has perfect state transfer between the two adjacent vertices of K_2 if $\Delta = \sqrt{\tilde{k}^2 + 8n}$ is an integer and $\tilde{k}, \Delta \equiv 0 \pmod 8$.*

Proof Let $\mathcal{G} = K_2 + G$. By Theorem 12, since there is perfect state transfer between the two vertices of K_2 at time $t^\star = (2\mathbb{Z}+1)\frac{\pi}{2}$, we have

$$\langle b|e^{-it^\star A_{\mathcal{G}}}|a\rangle = \langle b|e^{-it^\star A_G}|a\rangle + \frac{e^{-it^\star}}{2}\left\{e^{-it^\star\tilde{k}/2}\left[\cos\left(\frac{t^\star\Delta}{2}\right) + i\left(\frac{\tilde{k}}{\Delta}\right)\sin\left(\frac{t^\star\Delta}{2}\right)\right] - 1\right\}. \tag{66}$$

To achieve perfect state transfer in $\mathcal{G}$, it suffices to require

$$\frac{e^{-it^\star}}{2}\left\{e^{-it^\star\tilde{k}/2}\left[\cos\left(\frac{t^\star\Delta}{2}\right) + i\left(\frac{\tilde{k}}{\Delta}\right)\sin\left(\frac{t^\star\Delta}{2}\right)\right] - 1\right\} = 0 \tag{67}$$

or equivalently, $\cos(t^\star\tilde{k}/2)\cos(t^\star\Delta/2) = 1$ with $\Delta \in \mathbb{Z}$. Thus, it is sufficient to choose $\tilde{k}, \Delta \equiv 0 \pmod 8$ given $t^\star \in (2\mathbb{Z}+1)\frac{\pi}{2}$. □

Remark: The above corollaries complement the results of Bose *et al.* [2] on $K_2 + K_{m-2}$ and $\overline{K}_2 + K_{m-2}$ in the XYZ (Laplacian) interaction model. They showed that K_m has no perfect state transfer, but if we delete the edge (a, b) then there is perfect state transfer between a and b.

6 Conclusion

In this work, we studied perfect state transfer on quantum networks represented by graphs in the XY (adjacency) interaction model. Prior to our work, the only unweighted graphs known to have perfect state transfer were the cube-like networks [3], the Cartesian product $P_3^{\oplus n}$ of paths of length three [6], the path-like layered graph of diameter n [7], and the integral circulant graphs $\mathrm{ICG}_n(\{1, n/2^b\})$, for $n \equiv 0 \pmod 8$ and $b \in \{1, 2\}$ [4]. We described constructions of new families of graphs with perfect state transfer using graph operators which preserve this property. More specifically, we used the graph-theoretic join and its circulant generalizations as well as the Cartesian product. Most of our results involved a *reduction* argument from the larger graph structure to the individual graphs with respect to the perfect state transfer property.

We generalized both the "hypercube" Cartesian product and the join graph operators by defining a so-called *circulant join* $G +_{\mathrm{C}} G$ of two copies of a circulant graph G and connecting them using a circulant matrix C. This allowed us to interpolate between the above two interesting constructions and produced a graph operator that preserves the circulant property. From this construction, we derived new families of circulant graphs with perfect state transfer, namely, $\mathrm{ICG}_n(\{2, n/2^b\} \cup Q)$, $b \in \{1, 2\}$, where Q is a subset of the odd divisors of n. This expanded the class of circulant graphs known to have perfect state transfer (see [5, 4]).

Then, we showed that the Cartesian product of different perfect state transfer graphs has perfect state transfer provided all of these graphs share the same transfer time. This generalized previous results for paths of length two and three [6]. For the n-fold self-join, we showed that the existence of perfect state transfer on G^{+n} can be reduced to its existence in G along with other conditions. These observations allowed us to construct new families of graphs with perfect state transfer, for example $\mathrm{ICG}_n(\{1, n/2\})^{+m}$ or $\bigoplus_{k=1}^{m} G_k$, where G_k is any of the known integral circulant and hypercubic graphs with perfect state transfer, for any integer m.

Finally, we considered the join $G+H$ of two arbitrary regular graphs. Again, we reduced the existence of perfect state transfer on $G + H$ to its existence in G along with some conditions on the sizes and degrees of the two graphs. From this reduction, we constructed an interesting double-cone family of graphs $\overline{K}_2 + G$ which are non-periodic but with perfect state transfer. This answered one of the main questions posed in Godsil [9]. This also complemented results in Bose *et al.* [2] on perfect state transfer in the complete graphs K_m with a missing edge. Their results were stated in the XYZ (Laplacian) model while our results hold in the XY (adjacency) model.

It seems plausible that there is a characterization of perfect state transfer in integral circulants $G(n, \bigcup_{d \in D} G_n(d))$, for any D, using the join operator. This will complement the characterization of unitary Cayley graphs where $D = \{1\}$ [5]. It would also be interesting to consider perfect state transfer in weighted graphs (see [6]), especially on unweighted graphs which are known to lack the property. Finally, we find it curious that most of the graphs known with perfect state transfer achieve this at time $t^\star = (2\mathbb{Z} + 1)\pi/2$. This is true for the cube-like graphs [3] and for the integral circulants [4]. The lone exceptions are paths of length three [6] and the double-cones $\overline{K}_2 + G$, for suitable chosen regular graphs G.

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