WHEN DO TOEPLITZ AND HANKEL OPERATORS COMMUTE ?

RUBÉN A. MARTÍNEZ-AVENDAÑO

We completely classify all Toeplitz and Hankel operators which commute; namely, we prove that that a non–trivial Hankel operator and a non-trivial Toeplitz operator commute if and only if the Hankel operator has symbol $z\psi$, where ψ is the symbol of the Toeplitz operator, and ψ is an affine function of the characteristic function of certain "anti-symmetric" sets of the unit circle.

A Hankel operator H is a bounded linear operator on a separable Hilbert space such that $(He_m, e_n) = a_{n+m}$ for some sequence $\{a_n\}_{n=0}^{\infty}$ and some orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of the Hilbert space. This means that the matrix with respect to this orthonormal basis is constant along each diagonal perpendicular to the main one.

Analogously, a Toeplitz operator T is one for which $(Te_m, e_n) = b_{n-m}$ for some sequence $\{b_n\}_{n=-\infty}^{\infty}$. In this case the matrix is constant along all diagonals parallel to the main one.

It is natural to ask about the relationships between these two operators. Both operators have been studied for quite some time in many different ways (see the recent survey by Sarason [6] for an overview), but there is not a lot of knowledge about the commutant of a single Hankel operator. It is not even known whether the commutant contains a Toeplitz operator. In this note we will completely answer that question; that is, we will determine exactly when a Toeplitz operator and a Hankel operator commute.

1. Preliminaries.

We will assume that our operators act on the Hardy space $\mathbf{H}^2 = \mathbf{H}^2(S^1)$. Let us recall that this is the subspace of $\mathbf{L}^2 = \mathbf{L}^2(S^1, dm)$ (where dm is the normalized Lebesgue measure on the unit circle S^1) consisting of all functions with vanishing negative Fourier coefficients. This is a separable Hilbert space and the functions $e_n(z) = z^n$ for $n \geq 0$ form a basis, which we will refer to as the *standard* basis. Whenever a function f in $\mathbf{L}^2$ is also in $\mathbf{H}^2$ we say that f is *analytic*. Let us denote the space of essentially bounded functions on the

This paper is based on research which will form part of the author's University of Toronto Ph.D. thesis written under Prof. Peter Rosenthal. The author is partially supported by Mexico's Consejo Nacional de Ciencia y Tecnología (CONACyT).

unit circle (with respect to normalized Lebesgue measure) by $\mathbf{L}^\infty = \mathbf{L}^\infty(S^1)$ and the space of analytic bounded functions as $\mathbf{H}^\infty = \mathbf{L}^\infty \cap \mathbf{H}^2$.

The classical theorem of Nehari [3] (see Power [5] for a modern treatment) specifies when a sequence $\{a_n\}_{n=0}^\infty$ corresponds to a bounded Hankel operator. This is the case if and only if there exists a function $f \in \mathbf{L}^\infty$ such that a_n is the $(-n)$–th Fourier coefficient of f for $n \geq 0$. We then say f is a symbol for the Hankel operator H and we denote H by H_f. Notice that a symbol is not unique since for a function on $\mathbf{L}^\infty$ to be a symbol for a given Hankel operator only conditions on the non–positive Fourier coefficients are required. In fact, if $f, g \in \mathbf{L}^\infty$ and $f - g \in z\mathbf{H}^2$, then $H_f = H_g$.

Analogously, Hartman and Wintner [2] (see Brown and Halmos [1] for an alternative treatment) proved that a sequence $\{b_n\}_{n=-\infty}^\infty$ corresponds to a bounded Toeplitz operator if and only if there exists a function $g \in \mathbf{L}^\infty$ such that b_n is the n–th Fourier coefficient of g for all n. In this case we say g is the symbol of the Toeplitz operator T and we denote T by T_g. Note that in this case the symbol is unique.

For a function $f \in \mathbf{L}^2(S^1)$ we define $f^*(z) = \overline{f(\bar{z})}$, $\bar{f}(z) = \overline{f(z)}$ and $\tilde{f}(z) = f(\bar{z})$. Notice that the adjoint of a Hankel operator is also Hankel, and $H_f^* = H_{f^*}$. Similarly $T_g^* = T_{\bar{g}}$ for a Toeplitz operator. It is worth noticing that both Hankel and Toeplitz operators are linear with respect to their symbols.

There are two very important formulas that we will use throughout this note; both can be found in Power [4]. For f and $g \in \mathbf{L}^\infty$ we have

$$H_{\tilde{f}z} H_{gz} = T_{fg} - T_f T_g, \tag{1}$$

and

$$T_f H_g + H_{\tilde{f}z} T_{g\bar{z}} = H_{\tilde{f}g}. \tag{2}$$

The first of these formulas has been used extensively in the past to investigate relations between Hankel and Toeplitz operators but in this paper we will use mainly the second one.

2. Main part.

The "if" part of the following lemma is common knowledge among people who study Hankel operators and the proof is really easy. Unfortunately, we have not been able to find a reference for the "only if" part (although it is also undoubtedly known). We include the proof of both parts here.

Lemma 1. *Let H and T_g be a non-zero Hankel operator and a non-zero Toeplitz operator respectively. Then $T_g^* H = H T_{g^*}$ if and only if g is analytic.*

Notice that when $g(z) = z$ this condition is equivalent to an alternative definition of a Hankel operator: namely, H is Hankel if and only if $T_z^* H = H T_z$.

Proof. Let g have Fourier coefficients $a_k = \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) e^{-ik\theta} d\theta$. Let $H = H_f$, where f has Fourier coefficients $b_{-k} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ik\theta} d\theta$ (the minus sign on b_{-k} will save us from many more later on).

A straightforward calculation shows that

$$(T_g^* H_f z^n, z^m) = (H_f z^n, T_g z^m) = \sum_{k=0}^{\infty} b_{k+n} \bar{a}_{k-m} \tag{3}$$

and that

$$(H_f T_{g^*} z^n, z^m) = (T_{g^*} z^n, H_f^* z^m) = \sum_{k=0}^{\infty} \bar{a}_{k-n} b_{k+m}, \tag{4}$$

for $m \geq 0$ and $n \geq 0$.

First suppose g is analytic. Then, since $a_k = 0$ if $k < 0$, equation (3) becomes

$$(T_g^* H_f z^n, z^m) = \sum_{k=m}^{\infty} \bar{a}_{k-m} b_{k+n} = \sum_{s=0}^{\infty} \bar{a}_s b_{m+n+s}$$

and equation (4) becomes

$$(H_f T_{g^*} z^n, z^m) = \sum_{k=n}^{\infty} \bar{a}_{k-n} b_{k+m} = \sum_{s=0}^{\infty} \bar{a}_s b_{m+n+s}.$$

Since the right–hand sides of both equations are equal, $T_g^* H = H T_{g^*}$.

Conversely, assume $T_g^* H = H T_{g^*}$ and rewrite equation (3) as

$$(T_g^* H_f z^n, z^m) = \sum_{k=0}^{m-1} \bar{a}_{k-m} b_{k+n} + \sum_{k=m}^{\infty} \bar{a}_{k-m} b_{k+n} \tag{5}$$

and equation (4) as

$$(H_f T_{g^*} z^n, z^m) = \sum_{k=0}^{n-1} \bar{a}_{k-n} b_{k+m} + \sum_{k=n}^{\infty} \bar{a}_{k-n} b_{k+m}. \tag{6}$$

A change of variables as before shows that both second summands in the right hand sides of equations (5) and (6) are equal, and thus

$$\sum_{k=0}^{m-1} \bar{a}_{k-m} b_{k+n} = \sum_{k=0}^{n-1} \bar{a}_{k-n} b_{k+m} \tag{7}$$

when both $m, n > 0$. If $n = 0$ and $m > 0$ we obtain

$$\sum_{k=0}^{m-1} \bar{a}_{k-m} b_k = 0. \tag{8}$$

We will assume for the rest of this proof that $m > n$. The left–hand side of equation (7) can then be written as

$$\begin{aligned} \sum_{k=0}^{m-1} \bar{a}_{k-m} b_{k+n} &= \sum_{k=0}^{m-n-1} \bar{a}_{k-m} b_{k+n} + \sum_{k=m-n}^{m-1} \bar{a}_{k-m} b_{k+n} \\ &= \sum_{k=0}^{m-n-1} \bar{a}_{k-m} b_{k+n} + \sum_{s=0}^{n-1} \bar{a}_{s-n} b_{s+m}, \end{aligned}$$

but the last summand is equal to the right–hand side of equation (7). It follows that

$$\sum_{k=0}^{m-n-1} \bar{a}_{k-m} b_{k+n} = 0, \qquad \text{for } m > n > 0. \tag{9}$$

Now, clearly there must exist an integer n_0 such that $b_{n_0} \neq 0$ (otherwise H would be the zero operator).

If $n_0 = 0$ we can use equation (8) and proceed by strong induction. Indeed, equation (8) with $m = 1$ gives $\bar{a}_{-1}b_0 = 0$, which implies $a_{-1} = 0$ (recall $b_0 \neq 0$). Now assume that $a_{-1} = a_{-2} = \ldots = a_{-s} = 0$. Then equation (8) with $m = s+1$ becomes $\bar{a}_{-(s+1)}b_0 = 0$, which implies, of course, that $a_{-(s+1)} = 0$. That is, $a_{-s} = 0$ for all $s > 0$, or, in other words, g is analytic.

If $n_0 > 0$ we use equation (9) with $n = n_0$; that is,

$$\sum_{k=0}^{m-n_0-1} \bar{a}_{k-m}b_{k+n_0} = 0, \qquad \text{for } m > n_0. \tag{10}$$

We again use strong induction. If $m = n_0+1$ in equation (10) we obtain $\bar{a}_{-(n_0+1)}b_{n_0} = 0$, which in turn implies $a_{-(n_0+1)} = 0$ (recall again that $b_{n_0} \neq 0$). Now assume that $a_{-(n_0+1)} = a_{-(n_0+2)} = \ldots = a_{-(n_0+s)} = 0$. Then equation (10) with $m = n_0 + s + 1$ becomes $\bar{a}_{-(n_0+s+1)}b_{n_0} = 0$, which implies $a_{-(n_0+s+1)} = 0$. Thus $a_{-(n_0+s)} = 0$ for $s > 0$.

Now go back to equation (9). If we set $m = n_0 + 1$ we get

$$\sum_{k=0}^{n_0-n} \bar{a}_{k-n_0-1}b_{k+n} = \sum_{k=1}^{n_0-n} \bar{a}_{k-(n_0+1)}b_{k+n} = 0, \qquad \text{for } n_0 + 1 > n > 0, \tag{11}$$

since $a_{-(n_0+1)} = 0$ (as proven in the previous paragraph). If we set $n = n_0 - 1$ we get $\bar{a}_{-n_0}b_{n_0} = 0$ and thus $a_{-n_0} = 0$. Proceeding in this fashion we get $a_{-n_0} = a_{-(n_0-1)} = \ldots = a_{-2} = 0$.

Lastly, if in equation (8) we use $m = n_0 + 1$ we get $\bar{a}_{-1}b_{n_0} = 0$; thus $a_{-1} = 0$. Therefore $a_{-s} = 0$ for all $s > 0$; i.e., g is analytic.

□

Now we can prove a result concerning commutativity.

Theorem 2. *Let $\varphi \in \mathbf{L}^\infty$. Suppose that $\varphi = \tilde{\varphi}$ and that $HT_\varphi = T_\varphi H$ for a nonzero Hankel operator H. Then φ is a constant function (i.e., T_φ is a constant multiple of the identity).*

Proof. Since $\varphi = \tilde{\varphi}$, it follows that $(\overline{\varphi})^* = \varphi$. Also, as noted in Section 1, $T_\varphi = T^*_{\overline{\varphi}}$. Putting these two facts together, we see that $T_\varphi H = HT_\varphi$ is equivalent to $T^*_{\overline{\varphi}}H = HT_{(\overline{\varphi})^*}$. The latter equation implies, by Lemma 1, that $\overline{\varphi} \in \mathbf{H}^\infty$, and, since $\varphi = \tilde{\varphi}$, it follows that φ is constant.

□

We now prove the following

Theorem 3. *Let $f \in \mathbf{L}^\infty$. Then $T_f H = HT_f$ if and only if $T_{\tilde{f}}H = HT_{\tilde{f}}$.*

Note that this follows formally if we take the transpose of the product with respect to the standard basis. A more rigorous proof follows.

Proof. Let us define the anti–unitary involution V on $\mathbf{H}^2$ by $Vf = f^*$. It is easy to check that $VT_fV = T_{f^*}$, for $f \in \mathbf{L}^\infty$, and $VHV = H^*$ for any Hankel operator H. Clearly, $V^2 = I$.

Thus $T_fH = HT_f$ implies that $VT_fVVHV = VHVVT_fV$, which in turn implies that $T_{f^*}H^* = H^*T_{f^*}$. Taking adjoints we get $HT_{\tilde{f}} = T_{\tilde{f}}H$.

Applying this to $\tilde{f}$, it follows that $T_{\tilde{f}}H = HT_{\tilde{f}}$ implies $T_fH = HT_f$.

$\square$

The last two theorems allow us to get the following very useful corollary.

Corollary 4. *If $T_f H = H T_f$ for some function $f \in \mathbf{L}^\infty$ and a nonzero Hankel operator H, then $f + \tilde{f}$ is a constant function.*

Proof. If $T_f H = H T_f$ then, by Theorem 3, $T_{\tilde{f}} H = H T_{\tilde{f}}$. Both equations imply that $T_{f+\tilde{f}} H = H T_{f+\tilde{f}}$. Using Theorem 2 (since if $\varphi = f + \tilde{f}$ then $\varphi = \tilde{\varphi}$) it follows that $f + \tilde{f}$ is a constant function. $\square$

Theorem 5. *Let T_φ and H be a Toeplitz operator which is not a multiple of the identity operator and a nonzero Hankel operator respectively. If $T_\varphi H = H T_\varphi$ then $H = H_{\mu z \varphi}$, where μ is a complex number.*

Proof. Let φ have Fourier coefficients $a_k = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}) e^{-ik\theta} d\theta$. Let $H = H_\psi$, where ψ has Fourier coefficients $b_k = \frac{1}{2\pi}\int_0^{2\pi} \psi(e^{i\theta}) e^{-ik\theta} d\theta$.

For $m \geq 0$ and $n \geq 0$ we define (inspired by the classical paper of Brown and Halmos [1])

$$\begin{aligned} C_{n,m} &:= (H_\psi T_\varphi z^m, z^n) = (T_\varphi z^m, H_\psi^* z^n) \\ &= \sum_{k=0}^{\infty} a_{k-m} b_{-(k+n)} \end{aligned}$$

and

$$\begin{aligned} D_{n,m} &:= (T_\varphi H_\psi z^m, z^n) = (H_\psi z^m, T_\varphi^* z^n) \\ &= \sum_{k=0}^{\infty} b_{-(k+m)} a_{n-k}. \end{aligned}$$

For $m > 0$ and $n \geq 0$ we obtain

$$\begin{aligned} C_{n,m} &= \sum_{k=0}^{\infty} a_{k-m} b_{-(k+n)} \\ &= a_{-m} b_{-n} + \sum_{k=1}^{\infty} a_{k-m} b_{-(k+n)} \\ &= a_{-m} b_{-n} + \sum_{s=0}^{\infty} a_{s-(m-1)} b_{-(s+n+1)} \\ &= a_{-m} b_{-n} + C_{n+1,m-1} \end{aligned}$$

and

$$\begin{aligned} D_{n+1,m-1} &= \sum_{k=0}^{\infty} b_{-(k+m-1)} a_{n+1-k} \\ &= a_{n+1} b_{-(m-1)} + \sum_{k=1}^{\infty} b_{-(k+m-1)} a_{n+1-k} \\ &= a_{n+1} b_{-(m-1)} + \sum_{s=0}^{\infty} b_{-(s+m)} a_{n-s} \\ &= a_{n+1} b_{-(m-1)} + D_{n,m}. \end{aligned}$$

Since $H_\psi T_\varphi = T_\varphi H_\psi$, it follows that

$$\begin{aligned} C_{n,m} &= a_{-m}b_{-n} + C_{n+1,m-1} \\ &= a_{-m}b_{-n} + D_{n+1,m-1} \\ &= a_{-m}b_{-n} + a_{n+1}b_{-(m-1)} + D_{n,m} \\ &= a_{-m}b_{-n} + a_{n+1}b_{-(m-1)} + C_{n,m}\ , \end{aligned}$$

that is,

$$-a_{-m}b_{-n} = a_{n+1}b_{-m+1}\ , \qquad \text{for } m > 0 \text{ and } n \geq 0.$$

By Corollary 4, $\varphi + \tilde{\varphi}$ is constant, and thus $a_m + a_{-m} = 0$ if $m > 0$. Thus the previous equation becomes

$$a_m b_{-n} = a_{n+1}b_{-m+1}, \qquad \text{for } m > 0 \text{ and } n \geq 0. \tag{12}$$

Now, if there exists $n_0 \geq 0$ such that $b_{-n_0} = 0$, then equation (12) implies that $a_{n_0+1}b_{-m+1} = 0$ for $m > 0$, so either $a_{n_0+1} = 0$ or $b_{-m+1} = 0$ for all $m > 0$. But the latter would imply that $H = 0$, contradicting our hypothesis, so it follows that $a_{n_0+1} = 0$.

Similarly, if there exists $n_0 \geq 0$ such that $a_{n_0+1} = 0$, equation (12) implies that $a_m b_{-n_0} = 0$ for $m > 0$, so either $a_m = 0$ for all $m > 0$ or $b_{-n_0} = 0$. But the former would imply that φ is a constant (recall that $\varphi + \tilde{\varphi}$ is a constant); i.e., T_φ would be a multiple of the identity, contradicting our hypothesis, so it follows that $b_{-n_0} = 0$. Thus we have proven that $a_{n+1} = 0$ if and only if $b_{-n} = 0$ for $n \geq 0$.

Also, there must exist an integer $n_0 \geq 0$ such that $b_{-n_0} \neq 0$ (otherwise $b_{-n} = 0$ for all $n \geq 0$ which would imply, as before, that H is the zero operator). If we let $\lambda = \frac{b_{-n_0}}{a_{n_0+1}} \neq 0$, which is well defined by the previous paragraph, equation (12) implies that $\lambda a_m = b_{-m+1}$ for $m > 0$, or $\mu a_{-m} = b_{-m+1}$ for $m > 0$ and $\mu = -\lambda$.

This, of course, means that $\mu z\varphi - \psi \in z\mathbf{H}^\infty$, or equivalently, $H = H_\psi = H_{\mu z \varphi}$. This is what we wanted.

□

The conclusion of this theorem allows us to suppose, without loss of generality, that $H = H_{z\varphi}$ whenever $HT_\varphi = T_\varphi H$.

Theorem 6. *Let $\varphi \in \mathbf{L}^\infty$. If the Hankel operator $H = H_{z\varphi}$ commutes with the Toeplitz operator T_φ then $\varphi\tilde{\varphi}$ is a constant function.*

Proof. Since $H_{z\varphi}T_\varphi = T_\varphi H_{z\varphi}$, it follows by Theorem 3 that $H_{z\varphi}T_{\tilde{\varphi}} = T_{\tilde{\varphi}}H_{z\varphi}$. From equation (1) we get

$$H_{z\varphi}^2 = T_{\varphi\tilde{\varphi}} - T_{\tilde{\varphi}}T_\varphi,$$

and since, clearly, $H_{z\varphi}$ commutes with the left hand side and also commutes with the second term in the right–hand side, it must also commute with $T_{\varphi\tilde{\varphi}}$. But since $f = \varphi\tilde{\varphi}$ implies that $f = \tilde{f}$, it follows from Theorem 2 that $f = \varphi\tilde{\varphi}$ is a constant function.

□

We can summarize these theorems as follows.

Corollary 7. *If H is a nonzero Hankel operator such that $HT_\varphi = T_\varphi H$ for a Toeplitz operator T_φ not a multiple of the identity, then $H = \mu H_{z\varphi}$ for some $\mu \in \mathbb{C}$. In this case $\varphi + \tilde{\varphi}$ and $\varphi\tilde{\varphi}$ are constant functions.*

Of course, this last corollary does not imply that there are any non–trivial Toeplitz operators which commute with a Hankel operator. But it turns out that it does show us the way to an answer, as the following theorem states.

Theorem 8. *Let $\varphi \in \mathbf{L}^\infty$ be such that $\varphi + \tilde{\varphi}$ and $\varphi\tilde{\varphi}$ are constant functions. Then $H_{z\varphi}T_\varphi = T_\varphi H_{z\varphi}$.*

Proof. First of all, suppose that $\varphi + \tilde{\varphi} = c$ and $\varphi\tilde{\varphi} = d$ for some complex numbers c and d. From equation (2) we get

$$T_\varphi H_{z\varphi} + H_{z\tilde{\varphi}} T_\varphi = H_{z\varphi\tilde{\varphi}}.$$

But $\varphi\tilde{\varphi} = d$, so

$$T_\varphi H_{z\varphi} + H_{z\tilde{\varphi}} T_\varphi = H_{z\varphi\tilde{\varphi}} = H_{dz} = 0,$$

since $dz \in z\mathbf{H}^\infty$. Using $\varphi + \tilde{\varphi} = c$ we obtain

$$T_\varphi H_{z\varphi} + H_{z(c-\varphi)} T_\varphi = 0,$$

but $H_{cz} = 0$ as above. This means that

$$T_\varphi H_{z\varphi} - H_{z\varphi} T_\varphi = 0,$$

as desired.

□

What is so special about functions in $\mathbf{L}^\infty$ with the properties in the hypothesis of the last theorem ? The following lemma classifies them.

Lemma 9. *Let $\varphi \in \mathbf{L}^\infty$. Then $\varphi + \tilde{\varphi}$ and $\varphi\tilde{\varphi}$ are constant functions if and only if $\varphi(z) = a\chi_E(z) + b$, where a and $b \in \mathbb{C}$ and χ_E is the characteristic function of a measurable set $E \subset S^1$ such that $m(E^*\triangle E^c) = 0$. Here m is normalized Lebesgue measure on S^1, $E^* = \{z \in S^1 | \bar{z} \in E\}$, $E^c = \{z \in S^1 | z \notin E\}$, and $A\triangle B = (A \setminus B) \cup (B \setminus A)$.*

Proof. First the "if" part. If $\varphi(z) = a\chi_E(z) + b$ as in the statement of the lemma, then $\tilde{\varphi}(z) = a\chi_{E^*}(z) + b$ and

$$\begin{aligned}
\varphi(z) + \tilde{\varphi}(z) &= a(\chi_E(z) + \chi_{E^*}(z)) + 2b \\
&= a(\chi_{E\cup E^*}(z) - \chi_{E\cap E^*}(z)) + 2b \\
&= a(\chi_{S^1}(z) - \chi_{\emptyset}(z)) + 2b \\
&= a(1+0) + 2b \\
&= a + 2b,
\end{aligned}$$

because E^* and E^c coincide up to a set of measure 0.

Similarly

$$\begin{aligned}
\varphi(z)\tilde{\varphi}(z) &= (a\chi_E(z) + b)(a\chi_{E^*}(z) + b) \\
&= a^2\chi_E(z)\chi_{E^*}(z) + ab(\chi_E(z) + \chi_{E^*}(z)) + b^2 \\
&= a^2\chi_{E\cap E^*}(z) + ab + b^2
\end{aligned}$$

$$= \quad a^2 0 + ab + b^2,$$

so both are constant.

Conversely, suppose that $\varphi + \tilde{\varphi}$ and $\varphi\tilde{\varphi}$ are constant. We have two cases:

- Suppose $\varphi + \tilde{\varphi} = 1$ and $\varphi\tilde{\varphi} = 0$. Then

  $$\varphi = \varphi(\varphi + \tilde{\varphi}) = \varphi^2 + \varphi\tilde{\varphi} = \varphi^2,$$

  which implies that $\varphi = \chi_E$ for some measurable subset $E \subset S^1$. Since

  $$0 = \varphi(z)\tilde{\varphi}(z) = \chi_E(z)\chi_{E^*}(z) = \chi_{E\cap E^*}(z),$$

  it follows that $m(E \cap E^*) = 0$. Analogously,

  $$1 = \varphi(z) + \tilde{\varphi}(z) = \chi_E(z) + \chi_{E^*}(z) = \chi_{E\cup E^*}(z) - \chi_{E\cap E^*}(z) = \chi_{E\cup E^*}(z)$$

  so $m(E \cup E^*) = 1$. Thus $m(E^* \triangle E^c) = 0$ and the conclusion follows with $a = 1$ and $b = 0$.

- Suppose $\varphi + \tilde{\varphi} = c$ and $\varphi\tilde{\varphi} = d$. Let b satisfy the equation $b^2 - cb + d = 0$ and let $a = c - 2b$.

  If $a \neq 0$, then define $\psi(z) = \frac{\varphi(z)-b}{a}$. Then

  $$\psi(z) + \tilde{\psi}(z) = \frac{\varphi(z) - b}{a} + \frac{\tilde{\varphi}(z) - b}{a} = \frac{c - 2b}{a} = \frac{a}{a} = 1$$

  and

  $$\psi(z)\tilde{\psi}(z) = \left(\frac{\varphi(z) - b}{a}\right)\left(\frac{\tilde{\varphi}(z) - b}{a}\right) = \frac{d - bc + b^2}{a^2} = \frac{0}{a^2} = 0$$

  so, by the previous case, $\psi = \chi_E$ where E is as described above. Thus $\varphi(z) = a\psi(z) + b = a\chi_E(z) + b$.

  If $a = 0$ then let $\psi(z) = \varphi(z) - b$. Then

  $$\psi(z) + \tilde{\psi}(z) = (\varphi(z) - b) + (\tilde{\varphi}(z) - b) = c - 2b = 0$$

  and

  $$\psi(z)\tilde{\psi}(z) = (\varphi(z) - b)(\tilde{\varphi}(z) - b) = d - bc + b^2 = 0.$$

  Thus $\tilde{\psi} = -\psi$ and $-\psi^2 = 0$, which implies that $\psi = 0$; i.e., $\varphi(z) = b$, which is what we wanted.

□

To finish this note and to summarize, we put all of this together.

Theorem 10. *Let $\varphi \in \mathbf{L}^\infty$. The nonzero Hankel operator H commutes with the Toeplitz operator T_φ (not a multiple of the identity operator) if and only if $H = H_{\mu z\varphi}$, where $\mu \in \mathbb{C}\setminus\{0\}$ and $\varphi(z) = a\chi_E(z) + b$, with $a \neq 0$ and $b \in \mathbb{C}$, and $E \subset S^1$ such that $m(E^* \triangle E^c) = 0$.*

Acknowledgments. I would like to thank the referee for considerably shortening the proofs of Theorems 2 and 3.

References

[1] A. Brown, P. R. Halmos, *Algebraic properties of Toeplitz Operators*, J. Reigne Angew. Math., **213** (1963/1964), 89–102.

[2] P. Hartman, A. Wintner, *On the spectrum of Toeplitz matrices*, Amer. J. Math., **72** (1950), 359–366.

[3] Z. Nehari, *On bounded bilinear forms.* Ann. of Math., **65** (1957), 153–162.

[4] S. Power, *Hankel Operators on Hilbert Space*, Bull. London Math. Soc., **12** (1980), 422–442.

[5] S. Power, "*Hankel Operators on Hilbert Space*," Pittman Publishing, Boston, 1982.

[6] D. Sarason, *Holomorphic spaces: a brief and selective survey,* in "Holomorphic Spaces (Berkeley, CA, 1995)", 1–34, Math. Sci. Res. Inst. Publ. 33, Cambridge Univ. Press, Cambridge, 1998.

Department of Mathematics
University of Toronto
Toronto, Ont. M5S 3G3
Canada
ruben@math.toronto.edu

AMS subject classification: 47B35