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A Generalization of Hankel Operators

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We introduce a class of operators, called λ-Hankel operators, as those that satisfy the operator equation $S^*X - XS = \lambda X$, where S is the unilateral forward shift and λ is a complex number. We investigate some of the properties of λ-Hankel operators and show that much of their behaviour is similar to that of the classical Hankel operators (0-Hankel operators). In particular, we show that positivity of λ-Hankel operators is equivalent to a generalized Hamburger moment problem. We show that certain linear spaces of noninvertible operators have the property that every compact subset of the complex plane containing zero is the spectrum of an operator in the space. This theorem generalizes a known result for Hankel operators and applies to λ-Hankel operators for certain λ. We also study some other operator equations involving S. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Hankel and Toeplitz operators have both been studied for a long time. A Hankel operator on Hilbert space is one whose matrix representation with respect to an orthonormal basis is constant along the diagonals perpendicular to the main diagonal. A Toeplitz operator is one whose matrix representation is constant along the diagonals parallel to the main diagonal. For basic facts about Hankel and Toeplitz operators, the reader is referred to [5, 7, 16]. For a recent survey of Hankel and Toeplitz operators, see [19].

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These classes of operators can also be seen as solutions to some linear operator equations involving the unilateral forward shift S and its adjoint S^* (precise definitions will be given at the end of this section). In particular, it is well known that an operator H is Hankel if and only if $S^*H = HS$ and that an operator T is Toeplitz if and only if $S^*TS = T$.

Generalizations of these equations have been investigated. For example, Douglas [4] has studied the solutions to the equation $S^*XT = X$ for arbitrary contractions S and T. Pták [17] studied solutions to the equation $S^*X = XT$ when S and T are contractions. Power [14] studied simultaneous solutions to the equations $S^*X = XS$ for all $S \in \mathcal{S}$, where $\mathcal{S}$ is a commutative family of shifts.

In a different direction, Barría and Halmos [1] asked the following question: What are the solutions of the equation $S^*XS = \lambda X$ for λ an arbitrary complex number? This is a spectral problem, and was completely solved by Sun [21]. If $\lambda = 1$ the solutions of this equation are just the Toeplitz operators.

The objective of this paper is to study the type of equation proposed by Barría and Halmos, but for the case of Hankel operators. To do this, we first show that a lot of equations involving the shift have only trivial solutions. We will describe exactly the solutions of the equation $\lambda S^*X = XS$. Unfortunately, this is not a spectral problem. This consideration leads to the study of the equation $S^*X - XS = \lambda X$, solutions of which we will call λ-Hankel operators. These operators are a generalization of Hankel operators, and as such, they share some of their properties. We should mention that a different generalization (the "derived" Hankel matrices) has been studied by Heinig [8].

In this paper we study some properties of λ-Hankel operators, including invertibility, spectra, finite rank, and positivity. We include a theorem which partially describes the spectra of certain classes of operators with properties similar to the set of Hankel operators. It is also shown that positivity of λ-Hankel operators for some λ solves a generalization of the classical Hamburger moment problem.

We now introduce some basic definitions and notation. Our operators will act on a separable Hilbert space $\mathcal{H}$ which will usually be ℓ^2 or the Hardy space $\mathbf{H}^2 = \{f: f(z) = \sum_{n=0}^{\infty} a_n z^n, \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$ (see Duren [6] for the basic properties of Hardy spaces). As it is customary, we identify $\mathbf{H}^2$ with the space of its boundary functions. We denote the canonical basis in ℓ^2 by $\{e_n\}_{n=0}^{\infty}$, which we also identify with the functions $e_n \in \mathbf{H}^2$ defined by $e_n(z) = z^n$. Because of this identification, we say that a vector $f \in \mathcal{H}$ is a *polynomial* if it is a finite linear combination of elements of the basis $\{e_n\}$; that is, $f = a_0 e_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n e_n$, or, equivalently, $f(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ for some complex numbers $a_0, a_1, \ldots, a_{n-1}, a_n$.

The unilateral forward shift (or just forward shift) is the bounded linear operator $S: \ell^2 \to \ell^2$ defined to be

$$S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots).$$

The bounded operator S can be realized as multiplication by z on $\mathbf{H}^2$. Its adjoint, called the backward shift, is the operator S^* defined by

$$S^*(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, a_4, \ldots).$$

Notice that $Se_n = e_{n+1}$ for all n and $S^*e_0 = 0$ and $S^*e_n = e_{n-1}$ for $n \geqslant 1$.

Given two vectors f and $g \in \mathscr{H}$, we define the rank-one operator $f \otimes g$ by

$$(f \otimes g)\, h = (h, g)\, f, \qquad \text{for all} \quad h \in \mathscr{H}.$$

We will use throughout this paper the following easy-to-check properties of this operator: (i) $A(f \otimes g)\, B = (Af) \otimes (B^*g)$, (ii) $(f \otimes g)^* = g \otimes f$, and (iii) $\|f \otimes g\| = \|f\|\, \|g\|$.

Also, if A and B are two bounded operators, we write $A = B (\operatorname{mod} \mathscr{K})$ whenever $A - B$ is a compact operator.

2. SOLUTIONS OF SOME EQUATIONS INVOLVING THE SHIFT

As we mentioned above, Toeplitz and Hankel operators are characterized as solutions to the operator equations $S^*TS = T$ and $S^*H = HS$, respectively. In this section we investigate the solutions of equations involving different arrangements of the shift operator and its adjoint. In particular we show that a lot of these equations have no bounded solutions other than zero.

THEOREM 2.1. *Let A be a bounded operator with* $\operatorname{Ker} A = \{0\}$ *and B be any bounded operator. If X is a bounded solution of the operator equation $AX = BXS^*$, then $X = 0$.*

Proof. It suffices to prove that $Xe_n = 0$ for all n. We proceed by induction. For $n = 0$, $AXe_0 = BXS^*e_0 = 0$, so $Xe_0 = 0$. Assume now that $Xe_k = 0$ for some k. Then $AXe_{k+1} = BXS^*e_{k+1} = BXe_k = 0$, so $Xe_{k+1} = 0$. This completes the induction. ∎

As a corollary, we note that many of the modifications of the equations defining Toeplitz and Hankel operators have no nontrivial solutions.

COROLLARY 2.2. *If X is a bounded solution of any one of the equations* $X = SXS^*$, $X = S^*XS^*$, $SX = XS^*$, $SX = SXS^*$, *or* $SX = S^*XS^*$, *then* $X = 0$.

For completeness, we state the following observation.

THEOREM 2.3. *Let A be a bounded operator with dense range and B be any bounded operator. If X is a bounded solution of the operator equation* $SXB = XA$ *then* $X = 0$.

Proof. Take the adjoint of the equation and notice that if Ran A is dense, then Ker $A^* = \{0\}$. ∎

This leads to a solution of other modifications of the Toeplitz and Hankel equations.

COROLLARY 2.4. *If X is a bounded solution of any one of the equations* $X = SXS$, $XS^* = SXS$, *or* $XS^* = SXS^*$, *then* $X = 0$.

We note that some of the above results (for example if $SX = XS^*$ then $X = 0$) are well-known among people who study Hankel operators.

3. SOLUTIONS OF SOME EQUATIONS INVOLVING A PARAMETER

In [1], Barría and Halmos ask what the solutions of the operator equation $S^*XS = \lambda X$ are and how these eigenoperators relate to the case $\lambda = 1$ (Toeplitz operators). The equation was solved by Sun [21].

THEOREM 3.1 (S. Sun [21]). *Let $\lambda \in \mathbb{C}$. The operator equation $S^*XS = \lambda X$ has bounded solutions if and only if $|\lambda| \leqslant 1$. We then have that*

(i) *If $|\lambda| = 1$, all solutions are of the form $W_{\bar{\lambda}}T$, where T is a Toeplitz operator and W_λ is the diagonal unitary operator defined as $W_\lambda e_n = \bar{\lambda}^n e_n$ for all n.*

(ii) *If $|\lambda| < 1$, all solutions are compact operators of the form*

$$\sum_{n=0}^{\infty} \lambda^n((S^n f) \otimes e_n + e_n \otimes (S^n g))$$

for some f and $g \in \mathscr{H}$.

This suggests the study of the operator equation $\lambda S^*X = XS$, although in this case, this equation does not define a spectral problem. (Note that the modifications of the equations in Corollaries 2.2 and 2.4, by including multiplication by a nonzero λ on one side, still have no nonzero solutions.)

To completely solve the operator equation $\lambda S^* X = XS$, we first solve a different equation. The proof of the following theorem uses the techniques found in [21].

THEOREM 3.2. *Let A be a bounded operator with $\|A\| < 1$ and let S be the unilateral forward shift. Then X is a bounded solution of the operator equation $AX = XS$ if and only if X is compact and X has the form*

$$X = \sum_{n=0}^{\infty} (A^n \varphi) \otimes e_n$$

for some vector $\varphi \in \mathscr{H}$.

Proof. Assume first that

$$X = \sum_{n=0}^{\infty} (A^n \varphi) \otimes e_n,$$

for some vector $\varphi \in \mathscr{H}$. Then

$$\begin{aligned} AX &= \sum_{n=0}^{\infty} (A^{n+1} \varphi) \otimes e_n = \sum_{n=0}^{\infty} (A^{n+1} \varphi) \otimes (S^* e_{n+1}) \\ &= \sum_{n=0}^{\infty} ((A^{n+1} \varphi) \otimes e_{n+1})\, S = \sum_{n=1}^{\infty} ((A^n \varphi) \otimes e_n)\, S \\ &= \sum_{n=1}^{\infty} ((A^n \varphi) \otimes e_n)\, S + (\varphi \otimes e_0)\, S = \sum_{n=0}^{\infty} (A^n \varphi) \otimes e_n S \\ &= XS, \end{aligned}$$

since $(\varphi \otimes e_0)\, S = \varphi \otimes (S^* e_0) = 0$.

Now, assume that X satisfies the equation $AX = XS$. As is well known, $SS^* = I - (e_0 \otimes e_0)$ so that $SS^* = I (\mathrm{mod}\ \mathscr{K})$. This implies that $AXS^* = XSS^* = X (\mathrm{mod}\ \mathscr{K})$. Let $\|\cdot\|_e$ denote the essential norm. Then, if $\|X\|_e \neq 0$ (and since $\|A\|_e \leqslant \|A\| < 1$ and $\|S^*\|_e = 1$) we have

$$\|X\|_e \leqslant \|A\|_e\, \|X\|_e\, \|S^*\|_e < \|X\|_e,$$

which is impossible. Thus $\|X\|_e = 0$; that is, X is compact.

Define an operator-valued linear transformation $\hat{\tau}: \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ as

$$\hat{\tau}(Y) = AYS^*.$$

Since $\|A\| < 1$, it follows that $\|\hat{\tau}\| < 1$, so $I - \hat{\tau}$ is invertible and its inverse is given by

$$(I - \hat{\tau})^{-1} = \sum_{n=0}^{\infty} \hat{\tau}^n.$$

Now, $AX = XS$ implies that $AXS^* = XSS^* = X - X(e_0 \otimes e_0)$, so that $X - AXS^* = \varphi \otimes e_0$, where $\varphi = Xe_0$. This means that

$$(I - \hat{\tau})(X) = \varphi \otimes e_0,$$

so that

$$X = (I - \hat{\tau})^{-1} (\varphi \otimes e_0) = \sum_{n=0}^{\infty} \hat{\tau}^n(\varphi \otimes e_0) = \sum_{n=0}^{\infty} (A^n \varphi) \otimes e_n. \quad \blacksquare$$

Using this theorem, we can solve the operator equation $\lambda S^* X = XS$.

COROLLARY 3.3. *Let $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Then X is a bounded solution of the operator equation $\lambda S^* X = XS$ if and only if X is compact and is of the form*

$$X = \sum_{n=0}^{\infty} \lambda^n (S^{*n} \varphi) \otimes e_n$$

for some vector $\varphi \in \mathscr{H}$.

Proof. Let $A = \lambda S^*$ and use the previous theorem. $\blacksquare$

COROLLARY 3.4. *Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Then X is a bounded solution of the operator equation $\lambda S^* X = XS$ if and only if X is compact and is of the form*

$$X = \sum_{n=0}^{\infty} \left(\frac{1}{\lambda}\right)^n e_n \otimes (S^{*n} \varphi)$$

for some vector $\varphi \in \mathscr{H}$.

Proof. The operator X is a solution of the operator equation $\lambda S^*X = XS$, if and only if X^* satisfies $\bar{\lambda}X^*S = S^*X^*$, or equivalently, if and only if X^* satisfies

$$\frac{1}{\bar{\lambda}} S^*X^* = X^*S.$$

Now, taking $A = \frac{1}{\bar{\lambda}} S^*$ in the previous theorem, we obtain that X^* satisfies the previous equation if and only if X^* is compact and is of the form

$$X^* = \sum_{n=0}^{\infty} \left(\frac{1}{\bar{\lambda}}\right)^n (S^{*n}\varphi) \otimes e_n.$$

Taking adjoints proves the result. ∎

The only case that remains is $|\lambda| = 1$. In this case, the solutions turn out to be just unitary multiples of Hankel operators.

THEOREM 3.5. *Let $|\lambda| = 1$. Then X is a bounded solution of the equation $\lambda S^*X = XS$ if and only if $X = W_\lambda H$, where H is a Hankel operator and W_λ is the unitary diagonal operator defined as in Theorem* 3.1.

Proof. First of all, a calculation shows that $\bar{\lambda}SW_\lambda = W_\lambda S$. This implies that $\bar{\lambda}S = W_\lambda S W_\lambda^*$, so $\lambda S^* = W_\lambda S^* W_\lambda^*$.

Multiply the previous equation by X to obtain $\lambda S^*X = W_\lambda S^* W_\lambda^* X$. Since $\lambda S^*X = XS$ it follows that $XS = W_\lambda S^* W_\lambda^* X$, from which $W_\lambda^* XS = S^* W_\lambda^* X$; i.e., $W_\lambda^* X$ is a Hankel operator.

Conversely, let $X = W_\lambda H$ for some Hankel operator H. A calculation shows that $W_\lambda S^* = \lambda S^* W_\lambda$. Then $XS = W_\lambda HS = W_\lambda S^*H = \lambda S^* W_\lambda H = \lambda S^*X$. ∎

4. THE OPERATOR EQUATION $S^*X - XS = \lambda X$

More interesting operators arise from solving the equation $S^*X - XS = \lambda X$. Let us first point out that reversing the order of S and S^* results, again, in only trivial solutions: $SX - XS^* = \lambda X$ implies that $(S - \lambda) X = XS^*$ which only has the zero solution by Theorem 2.1.

It is worth pointing out that there are also no nontrivial solutions to the equations $SX - XS = \lambda X$ or $S^*X - XS^* = \lambda X$ as we will show presently. We first need a lemma, whose proof was suggested to us by Peter Rosenthal.

LEMMA 4.1. *Suppose $\lambda \neq 0$. Let $f \in \mathbf{H}^2$ such that $\|(S - \lambda)^n f\| \leqslant K$ for all n for a fixed number $K > 0$. Then $f = 0$.*

Proof. As is well known, for any $f \in \mathbf{H}^2$ we have

$$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta.$$

Let $\varepsilon > 0$ and define $A_\varepsilon = \{\theta \in [0, 2\pi) : |e^{i\theta} - \lambda| \geqslant 1 + \varepsilon\}$. In fact, choose ε in such a way that the measure of A_ε is nonzero. Then,

$$\begin{aligned} K^2 \geqslant \|(S-\lambda)^n f\|^2 &= \frac{1}{2\pi}\int_0^{2\pi} |e^{i\theta} - \lambda|^{2n} \, |f(e^{i\theta})|^2 \, d\theta \\ &\geqslant \frac{1}{2\pi}\int_{A_\varepsilon} |e^{i\theta} - \lambda|^{2n} \, |f(e^{i\theta})|^2 \, d\theta \\ &\geqslant \frac{1}{2\pi}(1+\varepsilon)^{2n} \int_{A_\varepsilon} |f(e^{i\theta})|^2 \, d\theta. \end{aligned}$$

But this is impossible unless $\int_{A_\varepsilon} |f(e^{i\theta})|^2 \, d\theta = 0$, which implies that $f(e^{i\theta}) = 0$ for $\theta \in A_\varepsilon$. But $\mathbf{H}^2$ functions cannot vanish in sets of nonzero measure unless they are identically zero. Thus $f = 0$. ∎

We can now prove the following theorem.

THEOREM 4.2. *Let $\lambda \neq 0$. If X is a bounded solution of the equation $SX - XS = \lambda X$ or of the equation $S^*X - XS^* = \lambda X$, then $X = 0$.*

Proof. Assume X is a bounded solution of $SX - XS = \lambda X$ for a nonzero λ. It then follows that $(S-\lambda)^n X = XS^n$ and thus that $(S-\lambda)^n Xe_0 = Xe_n$ for all n. Since X is bounded it follows that $\|(S-\lambda)^n Xe_0\| \leqslant \|X\|$. By the previous lemma, $Xe_0 = 0$. But this implies that $Xe_n = 0$ for all n. That is, $X = 0$.

If X is a solution of $S^*X - XS^* = \lambda X$, taking adjoints we obtain the previous case. ∎

Before solving the equation in the title of this section, we need to recall some notation and some known facts. If $b \in \mathbb{C}$, and $|b| < 1$, we can define $k_b \in \ell^2$ as

$$k_b = \sum_{n=0}^{\infty} \bar{b}^n e_n, \qquad \text{or, equivalently, as} \qquad k_b(z) = \frac{1}{1 - \bar{b}z} \in \mathbf{H}^2.$$

This is usually referred to as the *reproducing kernel* since $(g, k_b) = g(b)$ for each $g \in \mathbf{H}^2$. It is easy to see that

$$\|k_b\| = \frac{1}{\sqrt{1 - |b|^2}}.$$

The property of the reproducing kernel which is of interest for us is that

$$S^* k_b = \bar{b} k_b.$$

We first note the following theorem.

THEOREM 4.3. *Let $|\lambda| < 2$. Then the operator equation $S^* X - XS = \lambda X$ has nonzero solutions.*

Proof. If $|\lambda| < 2$, then it is always possible to choose a number $a \in \mathbb{C}$, with $|a| < 1$, such that $|a - \lambda| < 1$ (for example, choose $a = \lambda/2$). Then the rank-one operator $X = k_{\bar{a}} \otimes k_{a-\lambda}$ is a solution of the equation:

$$\begin{aligned} S^* X - XS &= (S^* k_{\bar{a}}) \otimes k_{a-\lambda} - k_{\bar{a}} \otimes (S^* k_{a-\lambda}) \\ &= a k_{\bar{a}} \otimes k_{a-\lambda} - (a - \lambda)\, k_{\bar{a}} \otimes k_{a-\lambda} \\ &= \lambda X. \quad \blacksquare \end{aligned}$$

Before going any further, let us realize that the problem of solving the equation $S^* X - XS = \lambda X$ is the problem of finding eigenoperators for the bounded operator-valued linear transformation $\tau \colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined as

$$\tau(X) = S^* X - XS.$$

The previous theorem tells us that the disk centred at the origin of radius 2 is contained in the spectrum of τ. A theorem of Rosenblum (see, for example, [18, p. 8]) tells us that $\sigma(\tau) \subset \sigma(S^*) - \sigma(S)$, so that $\sigma(\tau) \subset \{z \in \mathbb{C} : |z| \leqslant 2\}$ (since $\sigma(S^*) = \sigma(S) = \{z \in \mathbb{C} : |z| \leqslant 1\}$—see, for example, [18, p. 36]). In conclusion, $\sigma(\tau) = \{z \in \mathbb{C} : |z| \leqslant 2\}$. This means that there are no nonzero solutions of the equation $S^* X - XS = \lambda X$ when $|\lambda| > 2$.

Note. After a final version of this paper had been circulated, L. Robert-González, and independently, A. Feintuch and A. Markus, proved that there are no nonzero λ-Hankel operators if $|\lambda| = 2$.

It turns out, as we will see in the rest of this paper, that the solutions of $S^* X - XS = \lambda X$ have some properties like those of Hankel operators.

DEFINITION 4.4. We call X a *λ-Hankel operator* if $S^* X - XS = \lambda X$. Clearly, a 0-Hankel operator is just a Hankel operator.

For a fixed λ, the set of λ-Hankel operators forms a vector subspace of $\mathcal{B}(\mathcal{H})$. As we saw in the proof of Theorem 4.3, if $|\lambda| < 2$, for $|a| < 1$ and

$|a-\lambda|<1$, $k_{\bar{a}} \otimes k_{a-\lambda}$ is a λ-Hankel operator. If we choose a sequence of distinct complex numbers $\{a_n\}$ such that $|a_n|<1$ and $|a_n-\lambda|<1$, then, for any sequence of complex numbers $\{c_n\}$ such that $\sum_{n=0}^{\infty} |c_n| < \infty$, the operator

$$X = \sum_{n=0}^{\infty} c_n \sqrt{1-|a_n|^2} \sqrt{1-|a_n-\lambda|^2}\, k_{\bar{a}_n} \otimes k_{a_n-\lambda}$$

is a compact λ-Hankel operator (eigenspaces are norm closed). This shows that there are λ-Hankel operators of arbitrary rank: for example, choose the sequence $\{a_n\}$ in such a way as to make $\{a_n-\lambda\}$ a Blaschke sequence (always possible), and notice that the reproducing kernels are linearly independent.

The natural question of whether there are any noncompact λ-Hankel operators arises. To answer this, we will show that a subclass of λ-Hankel operators can have a spectrum that is arbitrary except that it must contain 0.

First, we mention some basic properties of λ-Hankel operators. All of them are known for Hankel operators.

THEOREM 4.5. *The adjoint of a λ-Hankel operator is a $(-\bar{\lambda})$-Hankel operator. A λ-Hankel operator is never invertible. Its kernel is an invariant subspace for S and the closure of its range is an invariant subspace for S^*. A nonzero λ-Hankel operator can be self-adjoint only when λ is purely imaginary.*

Proof. We prove only the case $\lambda \neq 0$. For the case $\lambda = 0$ the reader is referred to Power's book [16].

If X is a λ-Hankel operator, we get $S^*X^* - X^*S = -\bar{\lambda}X^*$ by taking adjoints, so X^* is a $(-\bar{\lambda})$-Hankel operator.

If X is a λ-Hankel operator, then $(S^*-\lambda)X = XS$. If X was invertible, it would mean that S and $S^*-\lambda$ are similar. But they are not (for example, compare their spectra!).

Let $f \in \operatorname{Ker} X$. Then $Sf \in \operatorname{Ker} X$, since $XSf = (S^*-\lambda)Xf = 0$. Thus $\operatorname{Ker} X$ is an invariant subspace for S.

Since X^* is a $(-\bar{\lambda})$-Hankel operator, $\operatorname{Ker} X^*$ is an invariant subspace of S. But this means that $(\operatorname{Ker} X^*)^{\perp}$ (which is the closure of the range of X) is an invariant subspace of S^*.

Suppose X was self-adjoint. Then X is both a λ-Hankel operator and a $(-\bar{\lambda})$-Hankel operator. But this means that $\lambda X = S^*X - XS = (-\bar{\lambda})X$, which implies that $\lambda + \bar{\lambda} = 0$; i.e., λ is purely imaginary. ∎

It is known (see [10]) that a Hankel operator can have any compact subset of the complex plane which contains the origin as its spectrum. This is part of a more general result about classes of operators that have certain properties in common with the class of Hankel operators.

THEOREM 4.6. *Let $\mathcal{F}$ be a vector space of noninvertible operators acting on a separable Hilbert space $\mathcal{H}$. Suppose $\mathcal{F}$ is closed in the strong operator topology of $\mathcal{B}(\mathcal{H})$. Let $\{\varphi_n\}$ be a sequence of linearly independent unit vectors in $\mathcal{H}$ with the property that $(\varphi_n, \varphi_m) \to 0$ as $n \to \infty$ and m fixed. Suppose that $\varphi_n \otimes \varphi_n \in \mathcal{F}$ for every n. Then, given any compact subset σ of the complex plane containing zero, there exists an operator Γ in $\mathcal{F}$ such that $\sigma(\Gamma) = \sigma$.*

Proof. The proof of this result is basically the same as the proof for Hankel operators given in [10]. We only sketch the main ideas, but the reader can fill in the details using [10]. For a complete proof, the reader is referred to [9].

If $\sigma = \{0\}$, then the operator $\Gamma = 0$ is in $\mathcal{F}$ and we are done. If not, then given the set σ, we choose a countable dense subset $\{b_n\}_{n=1}^{\infty}$ of $\sigma \setminus \{0\}$ (assume for a moment that this set is infinite). Assume we have constructed a sequence of operators $\Gamma_n \in \mathcal{F}$ of finite rank n, with the following properties (the construction will be done below):

(i) $\operatorname{Ran} \Gamma_n = (\operatorname{Ker} \Gamma_n)^{\perp}$; i.e., $\operatorname{Ran} \Gamma_n$ is a reducing subspace of Γ_n;

(ii) $\operatorname{Ran} \Gamma_n \subset \operatorname{Ran} \Gamma_{n+1}$;

(iii) the (nonzero) eigenvalues of Γ_n are exactly $b_1, b_2, \ldots, b_n$, with corresponding *normalized* eigenvectors $f_1^{(n)}, f_2^{(n)}, \ldots, f_n^{(n)}$;

(iv) for each n, the system $\{f_k^{(n)}\}_{k=1}^{n}$ is a Riesz basis (for its closed linear span) and its measure of nonorthogonality is strictly less than 2 (see [10] for the definition of the measure of nonorthogonality for a Riesz basis); and

(v) $\|f_k^{(n)} - f_k^{(n+1)}\| \leqslant 2^{-n}$ for $k = 1, 2, \ldots, n$,

and a property (vi), to be stated shortly.

Then, we will show that there exists an operator Γ with the desired spectrum. Before doing that, note that if σ consisted only of a finite number of nonzero points, say N, the construction of operators $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ satisfying properties (i)–(vi) would provide us with the desired operator Γ (just choose $\Gamma = \Gamma_N$). Thus we can restrict ourselves to the case when σ is an infinite set.

We can now show the existence of Γ. First notice that condition (v) implies that $\{f_k^{(n)}\} \to f_k$ as $n \to \infty$ for some $f_k \in \mathcal{H}$. It can be shown that the system $\{f_k\}_{k=1}^{\infty}$ is also a Riesz basis (for its closed linear span), and its measure of nonorthogonality is at most 2.

Define an operator Γ on $\mathscr{H}$ by

$$\Gamma f_k = b_k f_k, \qquad \text{for each} \quad k \geqslant 1,$$

and

$$\Gamma f = 0 \qquad \text{when } f \text{ is orthogonal to all the } \{f_k\}.$$

Then, it can be seen that $\Gamma_n \to \Gamma$ in the strong operator topology and thus $\Gamma \in \mathscr{F}$. Also, it is clear that $\sigma(\Gamma) = \overline{\{b_k\}_{k=1}^{\infty} \cup \{0\}} = \sigma$.

We now need to construct a sequence of operators with properties (i)–(vi). For every n, we define $\Gamma_{n,\tau}$ as

$$\Gamma_{n,\tau} = \sum_{k=1}^{n} b_k(1+t_k)\, \varphi_{s_k} \otimes \varphi_{s_k},$$

for $\tau = (t_1, t_2, ..., t_n) \in \mathbb{R}^n$ and an increasing sequence $\{s_k\}$ of integers to be chosen later. Our operator Γ_n will be $\Gamma_{n,\tau^{(n)}}$ for some fixed $\tau^{(n)} = (t_1^{(n)}, t_2^{(n)}, ..., t_n^{(n)}) \in \mathbb{R}^n$.

Let $\Lambda^{(n)}(\tau) = (\lambda_1^{(n)}(\tau), \lambda_2^{(n)}(\tau), ..., \lambda_n^{(n)}(\tau))$ be the nonzero eigenvalues of the operator $\Gamma_{n,\tau} \in \mathscr{F}$. We can now state condition (vi).

(vi) the Jacobian $\frac{d\Lambda^{(n)}}{d\tau} = \left(\frac{\partial \lambda_j^{(n)}}{\partial t_k}\right)_{j,k=1}^{n}$ is nonsingular at $\tau = \tau^{(n)}$.

Clearly, the ordering of eigenvalues is not essential for this condition to make sense. It is natural for our purposes to order the eigenvalues in such a way that $\lambda_k^{(n)}(\tau^{(n)}) = b_k$. That the Jacobian is well defined can also be checked.

We proceed to construct the operators by induction. If $n = 1$, we pick $s_1 = 1$ and $t_1^{(1)} = 0$. Then $\Gamma_1 = b_1 \varphi_1 \otimes \varphi_1 = \Gamma_{1,\tau^{(1)}}$ satisfies conditions (i)–(vi).

Let us suppose that we have constructed vectors $\tau^{(k)} \in \mathbb{R}^k$ and an increasing set of integers s_k, with $1 \leqslant k \leqslant n$, such that the operator $\Gamma_n = \Gamma_{n,\tau^{(n)}}$ satisfies conditions (i)–(vi). We must show that there is a vector $\tau^{(n+1)}$ and a positive number s_{n+1}, larger than s_k for $1 \leqslant k \leqslant n$, such that conditions (i)–(vi) are satisfied by the operator $\Gamma_{n+1,\tau^{(n+1)}}$.

Let $\tau = (t_1, t_2, ..., t_n) \in \mathbb{R}^n$, and let $\tilde{\tau} = (\tau, t_{n+1}) = (t_1, t_2, ..., t_n, t_{n+1}) \in \mathbb{R}^{n+1}$. For $s > s_n$ define the operator-valued function $(s, \tilde{\tau}) \mapsto G_{n+1}^{(s)}(\tilde{\tau})$ by

$$G_{n+1}^{(s)}(\tilde{\tau}) = \Gamma_{n,\tau} + b_{n+1}(1+t_{n+1})\, \varphi_s \otimes \varphi_s.$$

Clearly, $G_{n+1}^{(s)}(\tilde{\tau}) \in \mathscr{F}$. For $s > s_n$, define also the operator-valued function $(s, \tilde{\tau}) \mapsto H_{n+1}^{(s)}(\tilde{\tau})$ by

$$H_{n+1}^{(s)}(\tilde{\tau}) = \Gamma_{n,\tau} + b_{n+1}(1+t_{n+1})\, h_s \otimes h_s,$$

where h_s is the normalized projection of the vector φ_s onto the orthogonal complement of $E_n := \operatorname{Ran} \Gamma_n = \operatorname{Span} \{\varphi_j: 1 \leqslant j \leqslant n\}$. Note that $H^{(s)}_{n+1}((\tau^{(n)}, 0))$ has the required eigenvalues, but it might not be in $\mathscr{F}$.

Also notice that, for fixed $\tilde{\tau}$, the operators $H^{(s)}_{n+1}(\tilde{\tau})$, $s > s_n$, are unitarily equivalent to each other. Thus, there exists an operator $H_{n+1}(\tilde{\tau})$ (acting, say, on $\mathbb{C}^{n+1}$) and, for each s, there exists a unitary operator $U_s: \mathbb{C}^{n+1} \to \operatorname{Span}\{E_n, h_s\} = \operatorname{Span}\{h_s, \varphi_{s_j}: 1 \leqslant j \leqslant n\}$ such that

$$U_s^* H^{(s)}_{n+1}(\tilde{\tau})\, U_s = H_{n+1}(\tilde{\tau})$$

and such that the restriction of U_s^* to E_n does not depend on s (operators $H^{(s)}_{n+1}(\tilde{\tau})$ for fixed $\tilde{\tau}$ and different s coincide on E_n).

Using the fact that $(\varphi_n, \varphi_m) \to 0$ as $n \to \infty$ and m remains fixed, it can be checked that the operator-valued functions $\tilde{\tau} \mapsto U_s^* G^{(s)}_{n+1, \tilde{\tau}} U_s$ converge in $C^1(G)$-norm to $\tilde{\tau} \mapsto H_{n+1}(\tilde{\tau})$ (for every bounded domain G in $\mathbb{R}^n$) as $s \to \infty$.

Let $\Lambda^{(s)}(\tilde{\tau}) = (\lambda^{(s)}_1(\tilde{\tau}), \lambda^{(s)}_2(\tilde{\tau}), \ldots, \lambda^{(s)}_{n+1}(\tilde{\tau}))$ be the eigenvalues of $G^{(s)}_{n+1}(\tilde{\tau})$ restricted to $\operatorname{Span}\{E_n, \varphi_s\}$, and let $\Lambda(\tilde{\tau}) = (\lambda_1(\tilde{\tau}), \lambda_2(\tilde{\tau}), \ldots, \lambda_{n+1}(\tilde{\tau}))$ be the eigenvalues of $H_{n+1}(\tilde{\tau})$. Then it is not difficult to see that there exists a neighbourhood $\mathscr{U}$ of the point $(\tau^{(n)}, 0)$ such that $\Lambda^{(s)}(\cdot) \to \Lambda(\cdot)$ in $C^1(\mathscr{U})$ as $s \to \infty$.

The Jacobian $d\Lambda/d\tilde{\tau}$ is nonsingular at the point $(\tau^{(n)}, 0)$ since it can be easily seen to have the following form:

$$\begin{pmatrix} \left(\dfrac{\partial \lambda^{(n)}_k}{\partial t_j}\right)^n_{k, j=1} & \mathbb{O} \\ \mathbb{O} & b_{n+1} \end{pmatrix}.$$

The upper-left corner is nonsingular by the induction hypothesis (vi), and $b_{n+1} \neq 0$ by the choice of the set $\{b_k\}$.

Let $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$ and let $\Omega \subset \hat{\mathbb{N}} \times \mathbb{R}^{n+1}$ be a neighbourhood of the point $(\infty, (\tau^{(n)}, 0)) \in \hat{\mathbb{N}} \times \mathbb{R}^{n+1}$. Define the function $f: \Omega \to \mathbb{C}^{n+1}$ by

$$f(s, \tilde{\tau}) = \begin{cases} \Lambda^{(s)}(\tilde{\tau}), & \text{if } \ s \in \mathbb{N}; \\ \Lambda(\tilde{\tau}), & \text{if } \ s = \infty. \end{cases}$$

It can be seen that f is well defined for a neighbourhood Ω small enough. The previous remarks show that f is continuous and differentiable for Ω small enough and that $\partial f/\partial \tilde{\tau}$ is nonsingular at $(\infty, (\tau^{(n)}, 0))$. Thus we can apply an implicit function theorem (see, for example, [20, p. 278]) to the function f at the point $(\infty, (\tau^{(n)}, 0))$ and obtain that, for s large enough, there exists a vector $\hat{\tau}(s) \in \mathbb{R}^{n+1}$ such that

$$\Lambda^{(s)}(\tilde{\tau}(s)) = f(s, \hat{\tau}(s)) = f(\infty, (\tau^{(n)}, 0)) = \Lambda((\tau^{(n)}, 0)) = (b_1, b_2, ..., b_{n+1}).$$

Choosing s large enough, it can be seen that if we put $\tau^{(n+1)} = \tilde{\tau}(s)$ and we define $s_{n+1} = s$, it follows that $\Gamma_{n+1} = \Gamma_{n+1, \tau^{(n+1)}} = G_{n+1}^{(s_{n+1})}(\tau^{(n+1)})$ satisfies all the conditions (i)–(vi). This finishes the proof. ▮

This result can be applied to some subsets of λ-Hankel operators, as the following corollary shows. We also leave the following question unanswered: what other nontrivial classes of operators satisfy the hypothesis of the previous theorem?

COROLLARY 4.7. *Let $|\lambda| < 2$ be a purely imaginary number and let σ be any subset of the complex plane containing zero. Then there exists a λ-Hankel operator X such that $\sigma(X) = \sigma$.*

Proof. Let $\mathcal{F}$ be the set of λ-Hankel operators. We only need to check that $\mathcal{F}$ satisfies the conditions of the previous theorem. As mentioned before, the set of λ-Hankel operators is a vector subspace of $\mathcal{B}(\mathcal{H})$ and consists of noninvertible operators (see Theorem 4.5). Since the set of λ-Hankel operators is the set of solutions of the equation $S^*X - XS = \lambda X$, it follows that this set is closed in the strong operator topology (even in the weak operator topology!).

Let $\{a_n\}$ be a sequence in the open unit disk with imaginary part equal to $\lambda/(2i)$ and such that $|a_n| \to 1$ as $n \to \infty$ (clearly such a sequence always exists). Then $\overline{a_n} = a_n - \lambda$, which implies that $k_{\overline{a_n}} \otimes k_{\overline{a_n}}$ is in $\mathcal{F}$.

We can then define $\varphi_n = \sqrt{1 - |a_n|^2}\, k_{\overline{a_n}}$. Notice that $\varphi_n \otimes \varphi_n \in \mathcal{F}$ and

$$(\varphi_n, \varphi_m) = \frac{\sqrt{1 - |a_n|^2}\sqrt{1 - |a_m|^2}}{1 - a_n \overline{a_m}} \to 0, \quad \text{as} \quad n \to \infty.$$

Applying the previous theorem to the set $\mathcal{F}$ we obtain the desired result. ▮

In particular this result partially answers the question of existence of noncompact λ-Hankel operators.

COROLLARY 4.8. *Let $|\lambda| < 2$ be a purely imaginary number. Then there exist noncompact λ-Hankel operators.*

Proof. Compact operators can only have a discrete spectrum which accumulates, at most, at zero. ▮

It turns out that we can also get noncompact λ-Hankel operators if $|\lambda| = 1$. We first need a lemma whose proof is also inspired by [21].

LEMMA 4.9. *Let λ and μ be two complex numbers of modulus one. Suppose that X is a λ-Hankel operator. If $Y = W_{\bar{\mu}\lambda} X W_{\bar{\mu}\lambda}$, then Y is a μ-Hankel operator. (Here $W_{\bar{\mu}\lambda}$ is the diagonal unitary operator as defined in Theorem* 3.1.*)*

Proof. If X satisfies $S^*X - XS = \lambda X$, then $(\mu\bar{\lambda})\, S^*X - (\mu\bar{\lambda})\, XS = \mu X$. Left and right multiply by $W_{\bar{\mu}\lambda}$ to get

$$(\mu\bar{\lambda})\, W_{\bar{\mu}\lambda} S^* X W_{\bar{\mu}\lambda} - (\mu\bar{\lambda})\, W_{\bar{\mu}\lambda} X S W_{\bar{\mu}\lambda} = \mu W_{\bar{\mu}\lambda} X W_{\bar{\mu}\lambda}.$$

It is easy to verify that $(\mu\bar{\lambda})\, S W_{\bar{\mu}\lambda} = W_{\bar{\mu}\lambda} S$ and $(\mu\bar{\lambda})\, W_{\bar{\mu}\lambda} S^* = S^* W_{\bar{\mu}\lambda}$. Using these equalities, we obtain

$$S^* W_{\bar{\mu}\lambda} X W_{\bar{\mu}\lambda} - W_{\bar{\mu}\lambda} X W_{\bar{\mu}\lambda} S = \mu W_{\bar{\mu}\lambda} X W_{\bar{\mu}\lambda},$$

so $S^*Y - YS = \mu Y$. ∎

This lemma tells us that, for unimodular λ, it is sufficient to restrict ourselves to one choice of λ when studying λ-Hankel operators. This leads to the following theorem.

THEOREM 4.10. *Let μ be a complex number of modulus one. Then there exist noncompact μ-Hankel operators.*

Proof. We know that there exist noncompact λ-Hankel operators for λ purely imaginary (Corollary 4.7). In particular for $\lambda = i$ there are noncompact λ-Hankel operators. Let X be one of them. By the previous lemma $Y = W_{\bar{\mu}\lambda} X W_{\bar{\mu}\lambda}$ is a μ-Hankel operator, which cannot be compact (it it were, X would also be compact). ∎

Are there any noncompact λ-Hankel operators in the case where $|\lambda| < 2$ is not purely imaginary and is not of modulus one?

Note. After the final version of this paper had been circulated, the author and Peter Yuditskii found an affirmative answer to the previous question. Details are in [11].

5. OTHER PROPERTIES OF λ-HANKEL OPERATORS

In this section, we will study some other properties of λ-Hankel operators: in particular, symbols, when they are finite rank, and their relations to analytic and co-analytic Toeplitz operators.

If X is a λ-Hankel operator (not necessarily bounded), we call Xe_0 the *symbol* of X. The reason for this name is that, since $S^*X - XS = \lambda X$, we have that $(S^* - \lambda)X = XS$, so $(S^* - \lambda)^n X = XS^n$, which implies that

$$Xe_n = (S^* - \lambda)^n Xe_0;$$

i.e., X, as a densely defined operator on the polynomials, is uniquely determined by Xe_0. In fact, the following formula will be useful

$$(5.1)\qquad (Xe_m, e_n) = (XS^m e_0, e_n) = ((S^* - \lambda)^m Xe_0, e_n)$$
$$= (Xe_0, (S - \bar{\lambda})^m e_n) = \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} (Xe_0, S^k e_n)$$
$$= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} (Xe_0, e_{k+n}).$$

The reader should be warned that our definition of symbol differs slightly from the one used for classical Hankel operators. In the case of a classical Hankel operator H, if $He_0 = \psi$, then the function ϕ, defined as $\phi(z) = \psi(\bar{z})$, is a symbol of H.

As we saw before, a λ-Hankel operator is never invertible. It turns out that they are not even essentially invertible.

THEOREM 5.1. *Let X be a λ-Hankel operator. Then* $0 \in \sigma_e(X)$.

Proof. The case $\lambda = 0$ is known (see Power [16, p. 55]). If $\lambda \neq 0$, then we know that $(S^* - \lambda)X = XS$. If X was essentially invertible, $S^* - \lambda$ and S would be essentially similar. But, since $\sigma_e(S) = \sigma_e(S^*) =$ the unit circle, $S^* - \lambda$ and S cannot be essentially similar. ∎

A curious property of the set of λ-Hankel operators is that it is invariant when multiplied on the right by an analytic Toeplitz operator or on the left by a co-analytic Toeplitz operator.

THEOREM 5.2. *Let X be a λ-Hankel operator, T an analytic Toeplitz operator and T' a co-analytic Toeplitz operator. Then XT and $T'X$ are λ-Hankel operators.*

Proof. As is well known, T is an analytic Toeplitz operator if and only if $ST = TS$; and T' is a co-analytic Toeplitz operator if and only if $S^*T' = T'S^*$.

Then, $(XT)S = XST = (S^* - \lambda)(XT)$, so XT is λ-Hankel. Also $S^*(T'X) = T'S^*X = (T'X)(S + \lambda)$, so $T'X$ is a λ-Hankel operator. ∎

The classical theorem of Kronecker states that a Hankel matrix is of finite rank if and only if its symbol is a rational function. We have a similar theorem for λ-Hankel operators.

THEOREM 5.3. *Let X be a λ-Hankel operator with symbol $Xe_0 = \varphi \in \mathbf{H}^2$. Then X is of finite rank if and only if φ is a rational function.*

Proof. The columns of the matrix of X are just the vectors $(S^* - \lambda)^n \varphi$. That means that X is of finite rank at most N if and only if there exist constant numbers $c_0, c_1, c_2, \ldots, c_N$, not all zero, such that

$$\sum_{n=0}^{N} c_n (S^* - \lambda)^n \varphi = 0.$$

Let $d_k = \sum_{n=k}^{N} \binom{n}{k} (-\lambda)^{n-k} c_n$. It is not hard to see that $d_0 = d_1 = d_2 = \cdots = d_N = 0$ if and only if $c_0 = c_1 = c_2 = \cdots = c_N = 0$ (for example, do a calculation similar to the one in Lemma 6.2). But then the equation

$$\begin{aligned}\sum_{n=0}^{N} c_n (S^* - \lambda)^n \varphi &= \sum_{n=0}^{N} c_n \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} S^{*k} \varphi \\ &= \sum_{k=0}^{N} \left(\sum_{n=k}^{N} \binom{n}{k} (-\lambda)^{n-k} c_n \right) S^{*k} \varphi \\ &= \sum_{k=0}^{N} d_k S^{*k} \varphi\end{aligned}$$

implies that the vectors $\{S^{*k}\varphi\}_{k=0}^{N}$ are linearly dependent if and only if the vectors $\{(S^* - \lambda)^n \varphi\}_{n=0}^{N}$ are linearly dependent; i.e., the Hankel operator with symbol φ is of rank at most N if and only if X is of finite rank. But the Hankel operator with symbol φ is of finite rank if and only if φ is a rational function (see Partington [12, p. 37]). ∎

Notice that this theorem does not say that X is bounded; it says that its matrix is of finite rank. This of course brings us to an obvious question: when is X bounded? That is, for what symbols φ is the λ-Hankel operator X (densely defined on polynomials), with $Xe_0 = \varphi$, bounded? It is easy to formulate some necessary conditions on the symbol implied by the boundedness of the operator (see [9] for some of them). Is there a Nehari-type theorem? (For Nehari's theorem, see Power [16].)

The same question arises for compactness: for what symbols φ is the λ-Hankel operator X (densely defined on polynomials), with $Xe_0 = \varphi$, compact? Again, some sufficient conditions on the symbol that guarantee compactness are easy to find (for example, sufficient conditions on φ that guarantee that X is Hilbert–Schmidt are known [9]). Is there a Hartman-type theorem? (For Hartman's theorem, see Power [16].)

We partially answer the question of boundedness for positive λ-Hankel operators in the next section.

6. POSITIVITY OF λ-HANKEL OPERATORS

Among the properties that λ-Hankel operators share with Hankel operators is the characterization of positivity. Since λ-Hankel operators may only be self-adjoint when λ is purely imaginary, we restrict ourselves throughout this section to that case.

Let X be a (not necessarily bounded, but at least defined on polynomials) λ-Hankel operator. Assume there exists a nondecreasing function μ on the real line, thought of as a measure $d\mu$ on the real line throughout the rest of this paper, such that

$$(Xe_0, e_n) = \int_{\mathbb{R}} \left(t + \frac{\lambda}{2}\right)^n d\mu(t). \tag{6.1}$$

This expression completely characterizes the symbol of the operator X, and thus it also characterizes X (as a λ-Hankel operator densely defined on polynomials).

If f and g are polynomials, $f = \sum_{m=0}^{M} a_m e_m$ and $g = \sum_{n=0}^{N} b_n e_n$, then we have that (using Eq. (5.1))

$$\begin{aligned}
(Xf, g) &= \sum_{m=0}^{M} \sum_{n=0}^{N} a_m \overline{b_n} (Xe_m, e_n) \\
&= \sum_{m=0}^{M} \sum_{n=0}^{N} a_m \overline{b_n} \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} (Xe_0, e_{k+n}) \\
&= \sum_{m=0}^{M} \sum_{n=0}^{N} a_m \overline{b_n} \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} \int_{\mathbb{R}} \left(t + \frac{\lambda}{2}\right)^{k+n} d\mu(t) \\
&= \sum_{m=0}^{M} \sum_{n=0}^{N} a_m \overline{b_n} \int_{\mathbb{R}} \left(t + \frac{\lambda}{2}\right)^{n} \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} \left(t + \frac{\lambda}{2}\right)^{k} d\mu(t) \\
&= \int_{\mathbb{R}} \sum_{m=0}^{M} a_m \left(t - \frac{\lambda}{2}\right)^{m} \sum_{n=0}^{N} \overline{b_n} \left(t + \frac{\lambda}{2}\right)^{n} d\mu(t) \\
&= \int_{\mathbb{R}} f\left(t - \frac{\lambda}{2}\right) \overline{g\left(t - \frac{\lambda}{2}\right)} d\mu(t).
\end{aligned} \tag{6.2}$$

Therefore, if f is a polynomial we obtain

$$(6.3) \qquad (Xf, f) = \int_{\mathbb{R}} \left| f\left(t - \frac{\lambda}{2}\right)\right|^2 d\mu(t),$$

so $(Xf, f) \geqslant 0$ for all polynomials f. Thus X is positive.

Is the converse true? Namely, if X is a positive λ-Hankel operator, does there exist a nondecreasing function μ such that Eq. (6.1) holds? The answer is yes. The case for Hankel operators is well known (see Power [15]) and its solution is intimately related to the Hamburger moment problem. The Hamburger moment problem is a classical problem in the theory of moments that relates positivity of a Hankel matrix with the solution of a moment problem on the real line. For some very interesting results in the theory of moments, the reader should see [3]. For other operator-theoretic problems in the theory of moments, the reader should see [2].

We will use the solution of the Hamburger moment problem to answer the above question about λ-Hankel operators. We need some lemmas before we can prove Propositions 6.4 and 6.8, which will form the basic steps in the solution of a generalized Hamburger moment problem.

LEMMA 6.1. $\sum_{s=k}^{n} \binom{n}{s}\binom{s}{k}(-1)^{n-s} = \delta_{n,k}$, *where* $\delta_{n,k}$ *is the Kronecker delta.*

Proof. Apply the binomial theorem twice and change the order of the sums to obtain

$$\begin{aligned} x^n &= \sum_{s=0}^{n} \binom{n}{s} (x+1)^s (-1)^{n-s} \\ &= \sum_{s=0}^{n} \binom{n}{s} \sum_{k=0}^{s} \binom{s}{k} x^k (1)^{s-k} (-1)^{n-s} \\ &= \sum_{k=0}^{n} \left(\sum_{s=k}^{n} \binom{n}{s}\binom{s}{k} (-1)^{n-s} \right) x^k, \end{aligned}$$

which implies the desired result. ∎

Given a complex-valued sequence $\{\mu_n\}$, we define a sequence $\{m_n\}$ as

$$(6.4) \qquad m_n = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{\lambda}{2}\right)^{n-k} \mu_k.$$

It turns out that knowing $\{m_n\}$ allows us to recover $\{\mu_n\}$.

LEMMA 6.2. $\sum_{s=0}^{n}\binom{n}{s}\left(\frac{\lambda}{2}\right)^{n-s} m_s = \mu_n$.

Proof. Use the definition of $\{m_n\}$, interchange the sums, and use the previous lemma to see that

$$\begin{aligned}
\sum_{s=0}^{n}\binom{n}{s}\left(\frac{\lambda}{2}\right)^{n-s} m_s &= \sum_{s=0}^{n}\binom{n}{s}\left(\frac{\lambda}{2}\right)^{n-s}\sum_{k=0}^{s}\binom{s}{k}\left(-\frac{\lambda}{2}\right)^{s-k}\mu_k \\
&= \sum_{k=0}^{n}\sum_{s=k}^{n}\binom{n}{s}\binom{s}{k}\left(\frac{\lambda}{2}\right)^{n-s}\left(-\frac{\lambda}{2}\right)^{s-k}\mu_k \\
&= \sum_{k=0}^{n}\left(\sum_{s=k}^{n}\binom{n}{s}\binom{s}{k}(-1)^{n-s}\right)\left(-\frac{\lambda}{2}\right)^{n-k}\mu_k \\
&= \sum_{k=0}^{n}\delta_{n,k}\left(-\frac{\lambda}{2}\right)^{n-k}\mu_k \\
&= \mu_n. \quad \blacksquare
\end{aligned}$$

We need some notation that comes from the study of moment problems.

DEFINITION 6.3. Let λ be purely imaginary. We say that the sequence $\{v_n\}$ is *λ-positive*, if for all polynomials $p(x)=a_n x^n+\cdots+a_1 x+a_0$ such that $p(x+\frac{\lambda}{2})\geqslant 0$ for $x\in\mathbb{R}$, we have $\sum_{k=0}^{n} a_k v_k \geqslant 0$.

This agrees with the classical terminology when $\lambda=0$ (see Widder [22, p. 127]). We also agree to say that a sequence is positive whenever it is 0-positive. The following lemma relates λ-positivity to positivity.

PROPOSITION 6.4. *Let $\{\mu_n\}$ be a complex sequence and $\{m_n\}$ be defined by*

$$m_n=\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{\lambda}{2}\right)^{n-k}\mu_k.$$

If $\{\mu_n\}$ is λ-positive then $\{m_n\}$ is positive.

Proof. Suppose $q(x)=b_n x^n+\cdots+b_1 x+b_0$ is a polynomial and $q(x)\geqslant 0$ for all $x\in\mathbb{R}$. Define $p(x)=q(x-\frac{\lambda}{2})$. Then $p(x+\frac{\lambda}{2})=q(x)\geqslant 0$ for all $x\in\mathbb{R}$.
A calculation shows that

$$p(x)=\sum_{s=0}^{n}\left(\sum_{k=s}^{n}\binom{k}{s}\left(-\frac{\lambda}{2}\right)^{k-s} b_k\right)x^s,$$

so, since $p(x+\frac{\lambda}{2}) \geqslant 0$ for all $x \in \mathbb{R}$, it follows from the definition of λ-positivity that

$$\sum_{s=0}^{n} \left(\sum_{k=s}^{n} \binom{k}{s} \left(-\frac{\lambda}{2} \right)^{k-s} b_k \right) \mu_s \geqslant 0.$$

By changing the order of the sums, and recalling the definition of the sequence $\{m_n\}$, we can see that

$$\begin{aligned} 0 &\leqslant \sum_{s=0}^{n} \left(\sum_{k=s}^{n} \binom{k}{s} \left(-\frac{\lambda}{2} \right)^{k-s} b_k \right) \mu_s \\ &= \sum_{k=0}^{n} \left(\sum_{s=0}^{k} \binom{k}{s} \left(-\frac{\lambda}{2} \right)^{k-s} \mu_s \right) b_k \\ &= \sum_{k=0}^{n} m_k b_k, \end{aligned}$$

which implies $\{m_n\}$ is positive. ∎

We need the following definitions.

DEFINITION 6.5. If $f = (a_0, a_1, a_2, \ldots) \in \ell^2$, we define $f^* \in \ell^2$ by $f^* = (\overline{a_0}, \overline{a_1}, \overline{a_2}, \ldots)$. This is equivalent to defining $f^* \in \mathbf{H}^2$ as $f^*(z) = \overline{f(\bar{z})}$ for $f \in \mathbf{H}^2$.

DEFINITION 6.6. Let λ be a purely imaginary number. We define the *λ-moment operator M_λ associated to the λ-Hankel operator X* to be

$$M_\lambda(p) = (Xe_0, p^*),$$

where M_λ operates on polynomials p.

We need the following result.

LEMMA 6.7. *Let $|\lambda| < 2$ be purely imaginary. Suppose $q(x)$ is a polynomial with real coefficients and define $q_-(x) = q(x-\frac{\lambda}{2})$ and $q_+(x) = q(x+\frac{\lambda}{2})$. Then $M_\lambda(q_-^2) = (Xq_+, q_+)$.*

Proof. Since X is a λ-Hankel operator and λ is purely imaginary, it follows that

$$\left(S + \frac{\lambda}{2} \right)^* X = X \left(S + \frac{\lambda}{2} \right);$$

thus, by noticing that $(S+\frac{\lambda}{2})^n f=(z+\frac{\lambda}{2})^n f$ for all f, it follows that

$$\left(X\left(z+\frac{\lambda}{2}\right)^n, \left(z+\frac{\lambda}{2}\right)^m\right)=\left(Xe_0, \left(z+\frac{\lambda}{2}\right)^n \left(z+\frac{\lambda}{2}\right)^m\right),$$

for all n and m. It follows from this, and the fact that q has real coefficients, that

$$\left(Xq\left(z+\frac{\lambda}{2}\right), q\left(z+\frac{\lambda}{2}\right)\right)=\left(Xe_0, q\left(z+\frac{\lambda}{2}\right)q\left(z+\frac{\lambda}{2}\right)\right).$$

But $q_-^*=q_+$, so $(Xq_+, q_+)=(Xe_0, q_+^2)=(Xe_0, q_-^{*2})=M_\lambda(q_-^2)$. ∎

Clearly, M_λ is linear, and if $p(x)=x^n$, then $M_\lambda(p)=(Xe_0, e_n)$. If $\mu_n=(Xe_0, e_n)$, then it is clear that $\{\mu_n\}$ is λ-positive if and only if for all polynomials $p(x)=a_nx^n+\cdots+a_1x+a_0$ such that $p(x+\frac{\lambda}{2})\geqslant 0$ for $x\in\mathbb{R}$ we have that $M_\lambda(p)\geqslant 0$. Using this fact, we obtain the following theorem.

PROPOSITION 6.8. *Let X be a λ-Hankel operator and $\mu_n=(Xe_0, e_n)$. If X is a positive operator, then $\{\mu_n\}$ is a λ-positive sequence.*

Proof. Fix a polynomial p such that $p(x+\frac{\lambda}{2})\geqslant 0$ for all $x\in\mathbb{R}$. By the remark preceding the statement of the theorem, it suffices to show that $M_\lambda(p)\geqslant 0$.

Define f as $f(x)=p(x+\frac{\lambda}{2})$. Clearly f is a polynomial and has real coefficients (polynomials that take only real values on the real numbers have real coefficients). Since f is positive-valued on the reals, it follows, by a theorem of Pólya and Szegő (see [13, p. 77]), that f can be written as $f=g^2+h^2$ for some real polynomials g and h.

If we define f_-, g_-, and h_- as in the statement of the previous lemma, we then have $p=f_-=g_-^2+h_-^2$, so $M_\lambda(p)=M_\lambda(g_-^2)+M_\lambda(h_-^2)=(Xg_+, g_+)+(Xh_+, h_+)$. But this last expression is positive, since X is positive. It follows that $M_\lambda(p)\geqslant 0$. ∎

We can now answer the question about the existence of measures corresponding to positive λ-Hankel operators. Notice that this theorem extends the solution of the Hamburger moment problem to horizontal lines in the complex plane.

THEOREM 6.9. *A λ-Hankel operator X is positive if and only if there exists a nondecreasing function μ on the real line such that*

$$(Xe_0, e_n)=\int_{\mathbb{R}}\left(t+\frac{\lambda}{2}\right)^n d\mu(t),$$

for all n.

Proof. As we showed at the beginning of this section, the existence of the measure $d\mu$ satisfying the condition implies the positivity of X.

We prove the converse. Suppose X is positive. By Proposition 6.8, the sequence $\{\mu_n\}$, where $\mu_n = (Xe_0, e_n)$, is λ-positive. By Proposition 6.4 this implies that the sequence $\{m_n\}$, where m_n is defined as in Eq. (6.4), is positive. But the solution of the Hamburger moment problem implies that, for a positive sequence, there exists a nondecreasing function μ on the real line such that (see, for example, [22, p. 129])

$$m_n = \int_{\mathbb{R}} t^n \, d\mu(t).$$

By Lemma 6.2, it follows that

$$\begin{aligned}\mu_n &= \sum_{s=0}^{n} \binom{n}{s} \left(\frac{\lambda}{2}\right)^{n-s} \int_{\mathbb{R}} t^s \, d\mu(t) \\ &= \int_{\mathbb{R}} \left(t + \frac{\lambda}{2}\right)^n d\mu(t),\end{aligned}$$

by the binomial theorem. ∎

Consider the following example. If $d\mu$ is the atomic probability measure at $a \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} \left(t + \frac{\lambda}{2}\right)^n d\mu(t) = \left(a + \frac{\lambda}{2}\right)^n.$$

If this is the measure corresponding to a positive bounded λ-Hankel operator, we must have $|a + \frac{\lambda}{2}| < 1$. Let

$$\gamma = \sqrt{1 - \left|\frac{\lambda}{2}\right|^2}, \tag{6.5}$$

so that $|\gamma + \frac{\lambda}{2}| = 1$. Then, $|a + \frac{\lambda}{2}| < 1$ if and only if $a \in (-\gamma, \gamma)$. In this case, the atomic measure $d\mu$ corresponds to the rank one λ-Hankel operator

$$k_{\overline{a+\frac{\lambda}{2}}} \otimes k_{\overline{a+\frac{\lambda}{2}}}.$$

This suggests studying those measures that are supported on $(-\gamma, \gamma)$. We have the following theorem, analogous to the classical theorem of

Widom [23], which characterizes boundedness of the operator in terms of the speed of decay of the measure near the boundary of its support.

Before stating the theorem, we need the following definition.

DEFINITION 6.10. A positive Borel measure supported on the interval $(-\gamma, \gamma)$ is called a *Carleson measure* if $\mu(a, \gamma) = O(\gamma - a)$ and $\mu(-\gamma, -a) = O(\gamma - a)$ for $a \to \gamma^-$.

As we will see in the proof of the theorem, this definition agrees with the classical definition of a Carleson measure on the disk, as defined in [6, p. 157], when we view our measure as a measure on the disk.

THEOREM 6.11. *Let $|\lambda| < 2$ be a purely imaginary number. Let X be a positive λ-Hankel operator and suppose that the measure $d\mu$ corresponding to X is supported on $(-\gamma, \gamma)$. Then X is bounded if and only if $d\mu$ is a Carleson measure.*

Proof. Assume first that X is bounded. We need the following generalization of the calculation in Eq. (6.3).

Claim 1. If $|w| < 1$ then

$$(Xk_{\bar{w}}, k_{\bar{w}}) = \int_{-\gamma}^{\gamma} \left| \frac{1}{1 - w\left(t - \dfrac{\lambda}{2}\right)} \right|^2 d\mu(t).$$

Proof of Claim. By Eq. (6.3), if we define $k_{\bar{w}}^n = \sum_{k=0}^n w^k e_k$, we have

$$(Xk_{\bar{w}}^n, k_{\bar{w}}^n) = \int_{-\gamma}^{\gamma} \left| k_{\bar{w}}^n \left(t - \frac{\lambda}{2} \right) \right|^2 d\mu(t).$$

Clearly, $k_{\bar{w}}^n \to k_{\bar{w}}$ as $n \to \infty$, and, since X is bounded, we have

$$(Xk_{\bar{w}}, k_{\bar{w}}) = \lim_{n \to \infty} \int_{-\gamma}^{\gamma} \left| k_{\bar{w}}^n \left(t - \frac{\lambda}{2} \right) \right|^2 d\mu(t) = \int_{-\gamma}^{\gamma} \left| \frac{1}{1 - w\left(t - \dfrac{\lambda}{2}\right)} \right|^2 d\mu(t),$$

by Lebesgue's dominated convergence theorem. This establishes Claim 1. ▌

Now, for $0 < a < \gamma$, let

$$f = \frac{k_{\overline{a + \frac{\lambda}{2}}}}{\| k_{\overline{a + \frac{\lambda}{2}}} \|}.$$

It then follows from Claim 1 that

$$(Xf, f) = \int_{-\gamma}^{\gamma} \frac{1 - \left|a + \frac{\lambda}{2}\right|^2}{\left|1 - \left(a + \frac{\lambda}{2}\right)\left(t - \frac{\lambda}{2}\right)\right|^2} \, d\mu(t) \tag{6.6}$$

$$\geqslant \int_{a}^{\gamma} \frac{1 - \left|a + \frac{\lambda}{2}\right|^2}{\left|1 - \left(a + \frac{\lambda}{2}\right)\left(t - \frac{\lambda}{2}\right)\right|^2} \, d\mu(t).$$

We need to estimate the denominator of the integrand. We have that

$$\left|1 - \left(a + \frac{\lambda}{2}\right)\left(t - \frac{\lambda}{2}\right)\right|^2$$
$$\leqslant \max\left\{\left|1 - \left(a + \frac{\lambda}{2}\right)\left(a - \frac{\lambda}{2}\right)\right|^2, \left|1 - \left(a + \frac{\lambda}{2}\right)\left(\gamma - \frac{\lambda}{2}\right)\right|^2\right\},$$

for $t \in (a, \gamma)$, by noticing that the left-hand side is a quadratic polynomial in t and the coefficient of the quadratic term is positive. We then have two cases.

Case (i). $|1 - (a + \frac{\lambda}{2})(a - \frac{\lambda}{2})|^2 \geqslant |1 - (a + \frac{\lambda}{2})(\gamma - \frac{\lambda}{2})|^2$.

From Eq. (6.6) it then follows that

$$(Xf, f) \geqslant \int_{a}^{\gamma} \frac{1 - \left|a + \frac{\lambda}{2}\right|^2}{\left|1 - \left(a + \frac{\lambda}{2}\right)\left(a - \frac{\lambda}{2}\right)\right|^2} \, d\mu(t) = \frac{\mu(a, \gamma)}{1 - \left|a + \frac{\lambda}{2}\right|^2}.$$

But, since X is bounded and $\|f\| = 1$, we have

$$\mu(a, \gamma) \leqslant \|X\| \left(1 - \left|a + \frac{\lambda}{2}\right|^2\right).$$

It is easy to see that $1 - |a + \frac{\lambda}{2}| \leqslant \gamma - a$ and that $1 + |a + \frac{\lambda}{2}| \leqslant 2$. Then it follows from the previous equation that

$$\mu(a, \gamma) \leqslant 2\, \|X\|\, (\gamma - a), \qquad \text{for} \quad 0 < a < \gamma. \tag{6.7}$$

Case (ii). $|1-(a+\frac{\lambda}{2})(\gamma-\frac{\lambda}{2})|^2 \geqslant |1-(a+\frac{\lambda}{2})(a-\frac{\lambda}{2})|^2$.

From Eq. (6.6) it then follows that

$$(Xf, f) \geqslant \int_a^\gamma \frac{1-\left|a+\frac{\lambda}{2}\right|^2}{\left|1-\left(a+\frac{\lambda}{2}\right)\left(\gamma-\frac{\lambda}{2}\right)\right|^2} d\mu(t) = \frac{1-\left|a+\frac{\lambda}{2}\right|^2}{\left|1-\left(a+\frac{\lambda}{2}\right)\left(\gamma-\frac{\lambda}{2}\right)\right|^2} \mu(a, \gamma).$$

But, since X is bounded and $\|f\| = 1$, we have

$$\mu(a, \gamma) \leqslant \|X\| \frac{\left|1-\left(a+\frac{\lambda}{2}\right)\left(\gamma-\frac{\lambda}{2}\right)\right|^2}{1-\left|a+\frac{\lambda}{2}\right|^2}. \tag{6.8}$$

To finish this case, we need the following claim.

Claim 2. Suppose $|z_0| = 1$ and $\operatorname{Re} z_0 > 0$. Thus, there exists $h_0 > 0$ and $K > 0$ such that, for all $0 < h < h_0$, if $z = z_0 - h$, then $|1 - z\overline{z_0}|^2 \leqslant K(1-|z|^2)(z_0 - z)$.

Proof of claim. It is clear that if $z = z_0 - h$ then $|1 - z\overline{z_0}|^2 = |z_0 - z|^2 = h^2$. Let $z_0 = \cos\theta_0 + i \sin\theta_0$. Then, $1-|z|^2 = 1-|z_0-h|^2 = 2h\cos\theta_0 - h^2$.

Since $\cos\theta_0 > 0$, choose h_0 such that $0 < h_0 < 2\cos\theta_0$ and $K = 1/(2\cos\theta_0 - h_0)$. Then, if $0 < h < h_0$ and $z = z_0 - h$, it follows that $1/K \leqslant 2\cos\theta_0 - h$, which implies that $h^2 \leqslant K(2h\cos\theta_0 - h^2)\, h$, which implies that $|1 - z\overline{z_0}|^2 \leqslant K(1-|z|^2)(z_0 - z)$. ∎

Set $z_0 = \gamma + \frac{\lambda}{2}$. Since $\gamma > 0$ there exists a $h_0 > 0$ and a $K > 0$ for which the conclusion of Claim 2 holds. Choose $a_0 = \gamma - h_0$. Then, if $a_0 < a < \gamma$ and if $z = a + \frac{\lambda}{2}$, it follows that $z_0 - z = \gamma - a < h_0$, so $|1 - z\overline{z_0}|^2 \leqslant K(1-|z|^2)(z_0 - z)$ for some fixed K. But this implies that

$$\frac{\left|1-\left(a+\frac{\lambda}{2}\right)\left(\gamma-\frac{\lambda}{2}\right)\right|^2}{1-\left|a+\frac{\lambda}{2}\right|^2} \leqslant K(\gamma - a), \qquad \text{for} \quad a_0 < a < \gamma.$$

Combine this with Eq. (6.8) to get

$$\mu(a, \gamma) \leqslant K\, \|X\|\, (\gamma - a), \qquad \text{for} \quad a_0 < a < \gamma. \tag{6.9}$$

So in either Case (i) or Case (ii), Eqs. (6.7) and (6.9) imply that $\mu(a, \gamma) = O(\gamma - a)$ as $a \to \gamma^-$.

An analogous calculation shows that $\mu(-\gamma, -a) = O(\gamma - a)$ as $a \to \gamma^-$.

Now let us assume that μ is a Carleson measure. We will show that X is bounded. To achieve this, we will first show that μ is a Carleson measure in the classical sense (see Duren [6, p. 157] for the definition).

We consider $d\mu$ to be a measure on the unit disk supported on the set $(-\gamma, \gamma) + \frac{\lambda}{2} := \{t + \frac{\lambda}{2} : -\gamma < t < \gamma\}$ (just the translation of the interval $(-\gamma, \gamma)$ up by $\frac{\lambda}{2}$) instead of the interval $(-\gamma, \gamma)$.

Given $0 < h < 1$ and $\theta_0 \in [0, 2\pi]$, let S_h be a Carleson sector; i.e.,

$$S_h = \{z = re^{i\theta} : 1 - h \leqslant r < 1, \theta_0 \leqslant \theta \leqslant \theta_0 + h\}.$$

We must prove that $\sup_h \mu(S_h)/h < \infty$. It suffices to consider h small. If $S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2} = \emptyset$, then $\mu(S_h) = 0$. If $S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2} \neq \emptyset$, then

$$\mu(S_h) = \mu\left(S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2}\right) \leqslant C \text{ length of } \left(S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2}\right),$$

where C is a constant (coming from our definition of Carleson measure) and length of $(S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2})$ means the length of an interval inside the sector S_h which contains $S_h \cap (-\gamma, \gamma) + \frac{\lambda}{2}$. But the length of said interval is less than the perimeter of the sector S_h, which can be easily seen to be less than or equal to $4h$.[2] Thus

$$\frac{\mu(S_h)}{h} \leqslant 4C,$$

so

$$\sup_h \frac{\mu(S_h)}{h} < \infty;$$

i.e., $d\mu$ is a Carleson measure in the classical sense.

Now, as in Eq. (6.2), if f and g are polynomials, we have

$$(Xf, g) = \int_{-\gamma}^{\gamma} f\left(t - \frac{\lambda}{2}\right) \overline{g\left(t - \frac{\lambda}{2}\right)} \, d\mu(t).$$

But this implies that

$$(Xf, g) = \int_{-\gamma}^{\gamma} \overline{f^*\left(t + \frac{\lambda}{2}\right)} g^*\left(t + \frac{\lambda}{2}\right) d\mu(t).$$

[2] I thank Kobi Snitz for noticing this fact, which simplified the proof enormously.

Thinking of $d\mu$ as the measure in the disk $\mathbb{D}$ as before, this becomes

$$(Xf, g) = \int_{\mathbb{D}} \overline{f^*(z)}\, g^*(z)\, d\mu(z).$$

But since $d\mu$ is a Carleson measure (in the classical sense), a theorem of Carleson (see, for example [6, p. 157]) implies that

$$|(Xf, g)| \leqslant \int_{\mathbb{D}} |f^*(z)\, g(z)|\, d\mu(z) \leqslant C\, \|fg\|_1,$$

where $\|\cdot\|_1$ is the norm on the Hardy space $\mathbf{H}^1$.

But, as is well known (by the Cauchy–Schwarz inequality), $\|fg\|_1 \leqslant \|f\|\,\|g\|$, so

$$|(Xf, g)| \leqslant C\, \|f\|\,\|g\|;$$

i.e., X is bounded. ▌

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