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ESSENTIALLY HANKEL OPERATORS

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Abstract

The set of essentially Hankel operators is introduced and some of its properties are investigated. It is shown in particular that the set contains some operators not of the form 'Hankel plus compact', even when restricted to the class of essentially Hankel operators with trivial (Fredholm) index function.

1. *Introduction*

The essential commutant of the unilateral forward shift has been the object of study for several years. Barría and Halmos [**2**] brought much attention to this set, and in particular they asked for a way to characterize it. The essential commutant of the forward shift has sometimes been referred to as the set of essentially Toeplitz operators (see the definitions below).

In this paper we study the set of essentially Hankel operators. We will show that, although this set is not an algebra, is nevertheless a norm-closed self-adjoint vector space. It turns out that some sets related to the set of essentially Hankel operators (for example, its intersection with the set of essentially Toeplitz operators) are C*-algebras.

We also offer some examples of non-trivial essentially Hankel operators. Non-trivial means that they are not just compact perturbations of Hankel operators. Compare with polynomially compact operators (that is, essentially algebraic), which are compact perturbations of algebraic operators (see Olsen [**22**]), or with Riesz operators (that is, essentially quasinilpotent), which are compact perturbations of quasinilpotent operators (see West [**30**]).

We obtain a family of non-trivial essentially Hankel operators all with non-trivial (Fredholm) index function. This brings to mind the classical Brown–Douglas–Fillmore theorem [**4**]: essentially normal operators with trivial index function are compact perturbations of normal operators. We show that this is not the case for essentially Hankel operators; namely, we obtain an essentially Hankel operator with trivial index function which is not a compact perturbation of a Hankel operator.

Let us first introduce some basic definitions.

We will deal with bounded operators on the Hilbert space $\mathscr{H} = \mathbf{H}^2$, the Hardy space of the unit disk. We think of $\mathbf{H}^2$ as a subspace of $\mathbf{L}^2$ in the usual way. It is easy to see that $e_n(z) := z^n$, $n = 0, 1, 2, \ldots$, is an orthonormal basis for $\mathbf{H}^2$ and we will refer to it as the canonical basis. We will identify, as usual, functions in $\mathbf{H}^2$ and vectors in ℓ^2, namely: $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbf{H}^2$ is thought of as the vector $(a_0, a_1, a_2, a_3, \ldots) \in \ell^2$.

For a function $f \in \mathbf{L}^2$ we define $f^* \in \mathbf{L}^2$ as $f^*(z) = \overline{f(\bar{z})}$, and $\tilde{f} \in \mathbf{L}^2$ as $\tilde{f}(z) = f(\bar{z})$.

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The unilateral *forward shift* is the operator S on ℓ^2 defined as

$$S(a_0, a_1, a_2, a_3, \dots) = (0, a_0, a_1, a_2, a_3, \dots),$$

for each $(a_0, a_1, a_2, a_3, \dots) \in \ell^2$. This operator is clearly an isometry. It is easy to see that, on $\mathbf{H}^2$, S is just multiplication by the variable z. We will usually think of S as acting on $\mathbf{H}^2$.

Its adjoint, usually called the *backward shift*, can be seen to have the form

$$S^*(a_0, a_1, a_2, a_3, \dots) = (a_1, a_2, a_3, a_4, \dots),$$

for each $(a_0, a_1, a_2, a_3, \dots) \in \ell^2$.

A Toeplitz operator is one whose matrix representation with respect to the canonical basis is constant along the diagonals parallel to the main diagonal. It is well known (indeed, it can be taken as a definition) that an operator $T \in \mathscr{B}(\mathscr{H})$ is Toeplitz if and only if $S^*TS = T$. For basic facts about Toeplitz operators the reader should see [**10**, **18**, **28**]. Let us mention that a bounded operator T is Toeplitz if and only if there exists a function $\varphi \in \mathbf{L}^\infty$ such that

$$T = T_\varphi := PM_\varphi,$$

where P is the orthogonal projection from $\mathbf{L}^2$ onto $\mathbf{H}^2$ and M_φ is the operator of multiplication by φ on $\mathbf{H}^2$. The function φ is called the *symbol* of T_φ. It should be noted that $S = T_z$ and $S^* = T_{\bar{z}}$.

A Hankel operator is one whose matrix representation with respect to the canonical basis is constant along the diagonals perpendicular to the main diagonal. It can be seen that a bounded operator H is Hankel if and only if $S^*H = HS$. For the basic facts about Hankel operators, the reader is referred to [**25**, **26**, **28**]. The classical theorem of Nehari states that a Hankel operator H is bounded if and only if there exists a function $\psi \in \mathbf{L}^\infty$ such that

$$H = H_\psi := PJM_\psi,$$

where P and M_ψ are as before and J is the unitary *flip* operator: $Jf = \tilde{f}$ for each $f \in \mathbf{L}^2$. The function ψ is called a *symbol* of H_ψ.

Let us mention also that the classical theorem of Hartman (see [**26**]) classifies compactness of Hankel operators. A Hankel operator $H = H_{zf}$ is compact if and only if $f \in \mathbf{H}^\infty + \mathbf{C}$, where $\mathbf{H}^\infty := \mathbf{L}^\infty \cap \mathbf{H}^2$ and $\mathbf{C}$ is the set of continuous functions on the unit circle.

There are two very important formulas that relate Hankel and Toeplitz operators: both can be found in Power [**25**]. For f and $g \in \mathbf{L}^\infty$ we have

$$H_{z\tilde{f}}H_{zg} = T_{fg} - T_f T_g, \tag{1.1}$$

and

$$T_f H_{zg} + H_{z\tilde{f}} T_g = H_{z\tilde{f}g}. \tag{1.2}$$

For vectors f and $g \in \mathscr{H}$, we define the rank one operator $f \otimes g$ as $(f \otimes g)h = (h, g)f$, for every $h \in \mathscr{H}$. The following properties of $f \otimes g$ are easy to check: $A(f \otimes g)B = (Af) \otimes (B^*g)$ for any A and $B \in \mathscr{B}(\mathscr{H})$, and $\|f \otimes g\| = \|f\|\,\|g\|$.

We will also denote by $\mathscr{B}(\mathscr{H})$ the set of all bounded operators on our Hilbert space and by $\mathscr{K}$ the set of all compact operators on it. For A and $B \in \mathscr{B}(\mathscr{H})$ we will say that $A = B \pmod{\mathscr{K}}$ if $A - B \in \mathscr{K}$.

2. *Basic properties*

We can now recall the definition of an essentially Toeplitz operator.

DEFINITION 2.1. We say that a bounded operator T is *essentially Toeplitz* if $S^*TS - T$ is compact. We denote the set of all essentially Toeplitz operators by ess Toep.

Notice that since S is unitary in the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}$, it follows that T is essentially Toeplitz if and only if $ST - TS \in \mathscr{K}$; that is, if and only if T is in the essential commutant of the shift S.

The set ess Toep contains all Toeplitz operators (since if T is a Toeplitz operator then $S^*TS - T = 0$), and it contains all compact operators. In fact, it is clear that ess Toep is a C*-algebra, and it thus contains the C*-algebra generated by all Toeplitz operators. For more fascinating facts about ess Toep the reader should consult [**2**].

We can now define the essentially Hankel operators.

DEFINITION 2.2. We say that a bounded operator H is *essentially Hankel* if $S^*H - HS$ is compact. We denote the set of all essentially Hankel operators by ess Hank.

It will be useful sometimes to realize that $H \in$ ess Hank if and only if $H - SHS \in \mathscr{K}$ if and only if $S^*HS^* - H \in \mathscr{K}$, since S is unitary in the Calkin algebra.

Clearly, all Hankel operators are essentially Hankel. It is clear that ess Hank is a (norm-closed) vector subspace of $\mathscr{B}(\mathscr{H})$. In fact, it is even a self-adjoint set. Unfortunately, ess Hank is not an algebra, as the following proposition shows.

PROPOSITION 2.3. *Let $\varphi = \varphi^* \in \mathbf{L}^\infty$ such that $(1 - z^2)\varphi \notin \mathbf{H}^\infty + \mathbf{C}$. Then $H_{z\varphi}^2$ is not in* ess Hank.

Proof. Let $H = H_{z\varphi}$. Since $S^*H^2 - H^2S = HSH - HS^*H = H(S - S^*)H$, it follows that $H^2 \in$ ess Hank if and only if $HT_{z-\bar{z}}H \in \mathscr{K}$, but using Equation (1.2) we get

$$T_{z-\bar{z}}H_{z\varphi} = H_{z(\bar{z}-z)\varphi} - H_{z(\bar{z}-z)}T_\varphi$$

which implies that

$$HT_{z-\bar{z}}H \in \mathscr{K} \quad \text{if and only if} \quad H_{z\varphi}H_{z(\bar{z}-z)\varphi} - H_{z\varphi}H_{z(\bar{z}-z)}T_\varphi \in \mathscr{K}.$$

Since $(\bar{z} - z) \in \mathbf{H}^\infty + \mathbf{C}$, it follows by Hartman's theorem that $H_{z(\bar{z}-z)} \in \mathscr{K}$. Thus $H^2 \in$ ess Hank if and only if $H_{z\varphi}H_{z(\bar{z}-z)\varphi} \in \mathscr{K}$. By Equation (1.1) we get

$$H_{z\varphi}H_{z(\bar{z}-z)\varphi} = T_{\tilde{\varphi}(\bar{z}-z)\varphi} - T_{\tilde{\varphi}}T_{(\bar{z}-z)\varphi}.$$

Thus $H^2 \in$ ess Hank if and only if $T_{\tilde{\varphi}(\bar{z}-z)\varphi} - T_{\tilde{\varphi}}T_{(\bar{z}-z)\varphi} \in \mathscr{K}$, but by the Axler–Chang–Sarason–Volberg theorem [**1**, **29**] this last expression is compact if and only if

$$\mathbf{H}^\infty[\varphi^*] \cap \mathbf{H}^\infty[(\bar{z} - z)\varphi] \subset \mathbf{H}^\infty + \mathbf{C},$$

where $\mathbf{H}^\infty[\psi]$ denotes the uniformly closed subalgebra of $\mathbf{L}^\infty$ generated by $\mathbf{H}^\infty$ and ψ (sometimes called a Douglas algebra).

Clearly $\varphi = \varphi^* \in \mathbf{H}^\infty[\varphi^*]$. Since $(1-z^2) \in \mathbf{H}^\infty$ it follows that $(1-z^2)\varphi \in \mathbf{H}^\infty[\varphi^*]$. Also, $(\bar{z}-z)\varphi \in \mathbf{H}^\infty[(\bar{z}-z)\varphi]$. Since $z \in \mathbf{H}^\infty$ it follows that $(1-z^2)\varphi \in \mathbf{H}^\infty[(\bar{z}-z)\varphi]$. Combining these two results we obtain

$$(1-z^2)\varphi \in \mathbf{H}^\infty[\varphi^*] \cap \mathbf{H}^\infty[(\bar{z}-z)\varphi].$$

Assume that $H^2 \in \operatorname{ess\,Hank}$. Then

$$(1-z^2)\varphi \in \mathbf{H}^\infty[\varphi^*] \cap \mathbf{H}^\infty[(\bar{z}-z)\varphi] \subset \mathbf{H}^\infty + \mathbf{C}.$$

This is a contradiction, since $(1-z^2)\varphi \notin \mathbf{H}^\infty + \mathbf{C}$ by hypothesis. Therefore, $H^2 \notin \operatorname{ess\,Hank}$. □

It is not hard to find functions that satisfy the hypothesis of the proposition. As a matter of fact, it is known (and it is simple to see) that the condition $(1-z^2)\varphi \in \mathbf{H}^\infty + \mathbf{C}$ is equivalent to $H_{z\varphi} \in \operatorname{ess\,Toep}$. (According to [**25**] this seems to have been pointed out first by S. Axler. The result also can be found in [**6**].) Since for a self-adjoint Hankel operator H one can always find a bounded symbol φ such that $\varphi^* = \varphi$, to find an operator that satisfies the hypothesis of the proposition, one only needs to find a self-adjoint Hankel operator $H_{z\varphi} \notin \operatorname{ess\,Toep}$. (This is possible: if all self-adjoint Hankel operators were in ess Toep it would follow that all Hankel operators are in ess Toep, which is not true by Axler's observation above.)

While ess Hank is not an algebra, it does have some curious properties, reminiscent of the facts that the product of a Hankel operator and an analytic Toeplitz operator is a Hankel operator, that the product of a co-analytic Toeplitz operator and a Hankel operator is a Hankel operator, and that the product of two Hankel operators is always in the algebra of Toeplitz operators.

LEMMA 2.4. *Let H and $G \in$* ess Hank *and $T \in$* ess Toep. *Then $HT \in$* ess Hank, *$TH \in$* ess Hank *and $HG \in$* ess Toep.

Proof. Notice that $S^*HT - HTS = HST - HTS \pmod{\mathscr{K}}$, since $H \in \operatorname{ess\,Hank}$, but the right-hand side is compact since $ST - TS \in \mathscr{K}$. Analogously, one can show that $TH \in \operatorname{ess\,Hank}$.

Also, notice that $S^*HGS - HG = HSS^*G - HG \pmod{\mathscr{K}}$, since H and $G \in \operatorname{ess\,Hank}$, but $SS^* = I \pmod{\mathscr{K}}$ therefore $HSS^*G - HG$ is compact. □

In particular, this lemma implies the following corollary.

COROLLARY 2.5. *The set* ess Toep $\cap$ ess Hank *is a C*-algebra with no identity. Even more*, ess Hank *does not contain non-zero Toeplitz operators.*

Proof. It follows from Lemma 2.4 and the fact that ess Toep is a C*-algebra that if A and $B \in \operatorname{ess\,Toep} \cap \operatorname{ess\,Hank}$, then $AB \in \operatorname{ess\,Toep} \cap \operatorname{ess\,Hank}$. The rest of the properties of a C*-algebra are easy to check.

If T is a non-zero Toeplitz operator, then $S^*T - TS$ is also a non-zero Toeplitz operator, but non-zero Toeplitz operators are never compact so $T \notin \operatorname{ess\,Hank}$. In particular, the identity is not in ess Hank. □

It is known that ess Toep contains many Hankel operators (see [**5**, **6**]). The following weaker version of a theorem of Power [**24**, Theorem 1.3(i)] is now obvious.

COROLLARY 2.6. *Let A be an indexing set. Let $\{H_\alpha\}_{\alpha \in A}$ be a set of Hankel operators all of which are contained in* ess Toep. *Then the C*-algebra (with no identity) generated by $\{H_\alpha\}_{\alpha \in A}$ does not contain non-zero Toeplitz operators.*

We can now use a theorem of Power [**23**] to prove the following.

THEOREM 2.7. ess Toep + ess Hank *is a C*-algebra.*

Proof. We know that ess Toep is a C*-algebra. The vector space ess Hank is a norm-closed self-adjoint space which is also an (ess Toep)-bimodule by Lemma 2.4. Also, $(\text{ess Hank})^2 \subset \text{ess Toep}$ by Lemma 2.4. We can then use [**23**, Theorem 2] to conclude that ess Toep + ess Hank is a C*-algebra. □

We should point out that the previous theorem has been proved independently by Guo, Liu and Zhang in [**17**], using basically the same methods.

Following Barría and Halmos [**2**], we could define an *asymptotic Toeplitz operator in the Calkin algebra* as an operator T such that the sequence $S^{*n}TS^n$ converges in the Calkin algebra. Unfortunately, it is not hard to see that this class is the same as ess Toep. Analogously, one could define H to be an *asymptotic Hankel operator in the Calkin algebra* if $S^{*n}HS^{*n}$ converges in the Calkin algebra or if S^nHS^n converges in the Calkin algebra. In either case, it is easy to see that this class coincides with ess Hank.

There are definitions of (uniformly, strongly and weakly) asymptotic Hankel operators (see Feintuch [**11**, **12**]) which are different from the ones proposed here, but which do not seem to translate well to the Calkin algebra case. Is there a good definition for asymptotic Hankel in the Calkin algebra?

3. *The set* ess Hank *is not trivial*

As mentioned in [**2**], it is 'the experts' conviction' that ess Toep is a huge set. It is known to contain the C*-algebra generated by all Toeplitz operators as well as many other operators not contained in this C*-algebra. Since the C*-algebra generated by all Toeplitz operators contains $\mathscr{K}$, it then follows that ess Toep is not just the C*-algebra generated by the sum of Toeplitz operators and compact operators. (However see Feintuch [**12**] for a class of operators which is of this form!)

What happens for ess Hank? Clearly, all sums of Hankel operators and compact operators are in ess Hank. Are there any other types of operators in ess Hank? In other words, is it true that if $\Gamma \in \text{ess Hank}$ then $\Gamma = H + K$ for some bounded Hankel operator H and some compact operator K? The answer to this question turns out to be no, as we will show at the end of this section.

We first discuss some results, which, surprisingly, could be interpreted as evidence for an affirmative answer. Let us first formalize the question.

QUESTION. Does the space ess Hank consist only of compact perturbations of bounded Hankel operators?

It is known (see, for example [**26**]) that Hankel operators are never Fredholm, so any sum of a Hankel operator and a compact operator is not Fredholm either.

Thus the following lemma provides some (undoubtedly very weak) evidence that the answer to the question may be affirmative.

LEMMA 3.1. *If* $A \in \operatorname{ess\,Hank}$, *then* A *is not a Fredholm operator; that is*, $0 \in \sigma_{\mathrm{ess}}(A)$.

Proof. Let $A \in \operatorname{ess\,Hank}$. This means that $S^*A - AS = K$ for some compact operator K. Suppose that A was Fredholm. Then, since both S are S^* are Fredholm, we obtain the following relations [**8**, p. 354] for their Fredholm indices: $\operatorname{ind}(S^*A) = \operatorname{ind}(S^*) + \operatorname{ind}(A)$, and $\operatorname{ind}(AS) = \operatorname{ind}(A) + \operatorname{ind}(S)$.

However, since $S^*A = AS + K$, it follows that $\operatorname{ind}(S^*) = \operatorname{ind}(S)$. It is easy to check that $\operatorname{ind}(S) = -1$ and $\operatorname{ind}(S^*) = 1$, which is a contradiction. □

We need the following definition from [**2**].

DEFINITION 3.2. An operator A is said to be (*strongly*) *asymptotically Toeplitz* if $S^{*n}AS^n$ converges in the strong operator topology. If it converges, its limit is a Toeplitz operator T_φ and we say that φ is the 'symbol' of A.

This symbol coincides with the symbol of a Toeplitz operator but it does not coincide with the symbol of a Hankel operator.

It is known that all Hankel operators and all compact operators are (strongly) asymptotically Toeplitz (see [**2**]) and in fact, their symbol is 0. The following lemma then offers some more weak evidence of an affirmative answer to our question. It turns out, as we will show later, that not all operators in ess Hank are (strongly) asymptotically Toeplitz.

LEMMA 3.3. *If* $A \in \operatorname{ess\,Hank}$ *is* (*strongly*) *asymptotically Toeplitz, then its symbol is* 0.

Proof. Let $S^*A - AS = K$ and suppose that $S^{*n}AS^n$ converges in the strong operator topology to T_φ. Then $S^*AS = AS^2 + KS$, which implies that

$$S^{*(n+1)}AS^{n+1} = S^{*n}AS^nS^2 + S^{*n}KS^nS.$$

Taking limits in the strong operator topology, we obtain $T_\varphi = T_\varphi S^2 + 0$, since $S^{*n}KS^n$ converges in the strong operator topology to 0 (see [**2**]). This equation implies that $\varphi = 0$; that is, the symbol of A is zero. □

Another piece of weak evidence of an affirmative answer to the question is ahead, but first we need a lemma. This type of calculation is well known and has been used frequently (see, for example, Fialkow [**13**]), so we leave the details to the reader.

LEMMA 3.4. *Let* Q *be a compact operator such that*

$$C = -\sum_{n=0}^{\infty} S^{*n}QS^{*(n+1)}$$

converges in the uniform operator topology. Then C *is compact and* $S^*C - CS = Q$.

We obtain the following theorem.

THEOREM 3.5. *There is a dense linear manifold $\mathcal{Q}$ of $\mathcal{K}$ such that the following holds: if $S^*A - AS = Q$ with $Q \in \mathcal{Q}$, then A is a Hankel operator plus a compact operator.*

Proof. First, we will show that the linear manifold $\mathcal{Q}$, defined as those compact operators Q such that

$$\sum_{n=0}^{\infty} S^{*n} Q S^{*(n+1)}$$

converges in the uniform operator topology, is dense in $\mathcal{K}$. Let $Q = f \otimes g$ where f and $g \in \mathcal{H}$. Then, for any $N \geqslant M$,

$$\sum_{n=M}^{N} \|(S^{*n} f) \otimes (S^{n+1} g)\| = \sum_{n=M}^{N} \|S^{*n} f\| \, \|S^{n+1} g\|$$

$$= \|g\| \sum_{n=M}^{N} \|S^{*n} f\|.$$

This implies that $f \otimes g$ will be in $\mathcal{Q}$ if $\sum_{n=0}^{\infty} \|S^{*n} f\|$ converges. This happens for f in a dense subset of $\mathcal{H}$ (it happens for the polynomials, but it does not happen for all $f \in \mathcal{H}$). This means that a dense subset of the operators of rank one is in $\mathcal{Q}$. This implies that a dense subset of all finite rank operators is in $\mathcal{Q}$, ergo, a dense subset of the compact operators is in $\mathcal{Q}$.

Now, suppose that $S^*A - AS = Q$ with $Q \in \mathcal{Q}$. Then, if we define C to be

$$C = -\sum_{n=0}^{\infty} S^{*n} Q S^{*(n+1)},$$

then $S^*C - CS = Q$ and C is compact (by Lemma 3.4). This means that $S^*(A - C) - (A - C)S = 0$; that is, $A - C$ is a Hankel operator. □

We would like to point out that if we define $\tau : \mathcal{K} \longrightarrow \mathcal{K}$ as $\tau(X) = S^*X - XS$ then it can be seen that τ has dense range, which provides an alternative, but less explicit, proof of Theorem 3.5. Indeed, a theorem of Fialkow [**13**] (applied to $A = S^*$ and $B = S$) implies that $\tau : \mathcal{K} \longrightarrow \mathcal{K}$ has dense range if $\hat{\tau} : \mathbf{T} \longrightarrow \mathbf{T}$, defined as $\hat{\tau}(X) = SX - XS^*$, is injective ($\mathbf{T}$ is the space of trace class operators). However, it is easy to check (see for example [**20**, **21**]) that $SX = XS^*$ implies $X = 0$.

Using results from Fialkow [**13**] and Davis and Rosenthal [**9**] one can show that τ is not surjective, and therefore our previous theorem cannot be improved. In fact, using results from Fialkow [**14**] one can show that given a rank one operator $f \otimes g$, the equation $\tau(X) = f \otimes g$ does not always have a solution $X \in \mathcal{B}(\mathcal{H})$.

Of course, even though the operator τ is not surjective, the answer to our question could still be yes. The argument in the previous paragraph just says that $\tau(\mathcal{K}) \neq \mathcal{K}$ but the question has an affirmative answer if $\tau(\mathcal{B}(\mathcal{H})) \cap \mathcal{K} \subset \tau(\mathcal{K})$. As we will show later, the answer to the question is negative, thus $\tau(\mathcal{B}(\mathcal{H})) \cap \mathcal{K} \not\subset \tau(\mathcal{K})$.

Incidentally, it is obvious that $\tau(K) \subset \tau(\mathcal{B}(\mathcal{H})) \cap \mathcal{K}$, so the answer to our question implies that $\tau(\mathcal{K}) \subset \tau(\mathcal{B}(\mathcal{H})) \cap \mathcal{K} \subset \mathcal{K}$, where the first inclusion is proper. (The second inclusion is also proper: if $\tau : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is defined as above, then a theorem of Davis and Rosenthal [**9**] implies that τ is not surjective; a theorem of Fialkow [**13**] implies then that $\mathcal{K} \not\subset \tau(\mathcal{B}(\mathcal{H}))$; that is, $\tau(\mathcal{B}(\mathcal{H})) \cap \mathcal{K} \neq \mathcal{K}$.)

Before presenting the example of an operator in ess Hank which cannot be written in the form Hankel plus compact, we prove a theorem about Rhaly (or terraced) matrices.

DEFINITION 3.6. An infinite matrix of the form

$$R = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_1 & 0 & 0 & \cdots \\ a_2 & a_2 & a_2 & 0 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

for some complex sequence $\{a_n\}_{n=0}^{\infty} \in \ell^2$, is called a *Rhaly matrix* (or a *terraced matrix*). See [**19**, **27**] for more on this topic.

It is known that a Rhaly matrix determines a bounded operator if na_n is bounded [**19**], while a Rhaly matrix determines a compact operator if $na_n \to 0$ [**27**]. Both conditions are known not to be necessary. We have the following theorem.

PROPOSITION 3.7. *Let R be a bounded Rhaly matrix as in Definition 3.6. Then $R \in \operatorname{ess\,Toep}$ if and only if $R \in \operatorname{ess\,Hank}$.*

Proof. Define the infinite matrix A as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ a_2 - a_1 & 0 & 0 & 0 & \cdots \\ a_3 - a_2 & a_3 - a_2 & 0 & 0 & \cdots \\ a_4 - a_3 & a_4 - a_3 & a_4 - a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A calculation shows that

$$S^*RS - R = A + \operatorname{diag}(a_1 - a_0, a_2 - a_1, a_3 - a_2, \dots).$$

Since $\operatorname{diag}(a_1 - a_0, a_2 - a_1, a_3 - a_2, \dots)$ is compact, this implies that $R \in \operatorname{ess\,Toep}$ if and only if $A \in \mathcal{K}$.

Analogously, it can be shown that

$$S^*R - RS = A + \operatorname{diag}(a_1, a_2, a_3, \dots) + \operatorname{diag}(a_1, a_2, a_3, \dots)S^*.$$

Clearly $\operatorname{diag}(a_1, a_2, a_3, \dots)$ is compact, which implies that $R \in \operatorname{ess\,Hank}$ if and only if $A \in \mathcal{K}$. This concludes the proof. □

It is worth pointing out that one can easily show that, for a bounded Rhaly matrix R, $S^*RS - R$ is Hilbert–Schmidt if and only if $S^*R - RS$ is Hilbert–Schmidt.

We can now present a counterexample to our question.

THEOREM 3.8. *Let C be the Cesàro matrix; that is, the Rhaly matrix corresponding to the sequence $\{1/(n+1)\}_{n=0}^{\infty} \in \ell^2$. Then $C \in \operatorname{ess\,Hank}$ and C is not a compact perturbation of a bounded Hankel operator.*

Proof. A direct computation (see [**2**]) shows that $SC - CS$ is Hilbert–Schmidt, so that $C \in \operatorname{ess}\operatorname{Toep}$. By the previous proposition, $C \in \operatorname{ess}\operatorname{Hank}$.

Now, assume there is a bounded Hankel operator H and a compact operator K such that $C = H + K$. Take a point $\lambda \in \{z \in \mathbb{C} : |z - 1| < 1\}$. It is known [**16**] that $C - \lambda$ is Fredholm and it has Fredholm index $\operatorname{ind}(C - \lambda) = -1$.

Then, $C - \lambda = H - \lambda + K$ which implies that $H - \lambda$ is also Fredholm and $\operatorname{ind}(H - \lambda) = -1$. However, it is well known (and simple to prove) [**25**] that $\operatorname{ind}(H - \lambda) = 0$ for any bounded Hankel operator H whenever $H - \lambda$ is Fredholm, which is a contradiction. □

Let us point out the following from the previous proof. For an integer $n \geqslant 1$, we can define $A_n := (C - 1)^n - (-1)^n$, where C is the Cesàro matrix. It is clear that A_n is in $\operatorname{ess}\operatorname{Hank}$ (and also in $\operatorname{ess}\operatorname{Toep}$), since it is a polynomial function (with no constant term) of an operator in $\operatorname{ess}\operatorname{Hank} \cap \operatorname{ess}\operatorname{Toep}$. Also, $A_n + (-1)^n$ is Fredholm, since $C - 1$ is Fredholm.

Notice that $\operatorname{ind}(A_n + (-1)^n) = \operatorname{ind}((C-1)^n) = n \operatorname{ind}(C-1) = -n$, since $\operatorname{ind}(C-1) = -1$, as pointed out in the previous proof. Also, A_n^* is also in $\operatorname{ess}\operatorname{Hank}$ and $\operatorname{ind}(A_n^* + (-1)^n) = n$. Therefore, we can conclude that given any integer n, there exists an essentially Hankel operator Γ whose (Fredholm) index function has n in its range.

As mentioned in the introduction, the previous family of examples brings to mind the classical Brown–Douglas–Fillmore theorem: essentially normal operators with trivial index functions are compact perturbations of normal operators. The following example shows that this is not the case for the class of essentially Hankel operators.

Let A_H be the operator with infinite matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 1/2 & 0 & 0 & 0 & 1/4 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 2/4 & 0 & 0 & 0 & \cdots \\
0 & 1/2 & 0 & 0 & 0 & 3/4 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 3/4 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 2/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7/8 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 7/8 & & & \\
\vdots & \vdots & \vdots & & & & & & & &
\end{pmatrix}.$$

To be precise, the matrix is zero at all entries except those in the (n,m)-th place $(n, m \geqslant 0)$ if $n + m = 2^k$, for some positive integer k. At those places the value is $1 - |n - m|/2^k$.

To observe that A_H is the matrix of some bounded operator, notice that it is a symmetric matrix with non-negative entries and all rows and all columns (except the first row and first column) add up to 2. Then, a straightforward application of Schur's test (see, for example, [**18**, Problem 45]) implies that this is the matrix of a bounded operator (and in fact, it has norm less than or equal to 2).

Theorem 3.9. *Let A_H be the bounded operator with matrix as above. Then, $A_H \in$* ess Hank, *the index function of A_H is trivial and A_H is not a compact perturbation of a Hankel operator.*

Proof. The index function of A_H is trivial since it is self-adjoint. To see that A_H is in ess Hank, observe that the matrix of $S^*A_H - A_HS$ defines a Hilbert–Schmidt operator:

$$\begin{pmatrix} 0 & 1 & 0 & 1/2 & 0 & 0 & 0 & 1/4 & 0 & \cdots \\ -1 & 0 & 1/2 & 0 & 0 & 0 & 1/4 & 0 & 0 & \cdots \\ 0 & -1/2 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & \cdots \\ -1/2 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -1/4 & & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/8 & & \\ \vdots & \vdots & \vdots & & & & & & & \end{pmatrix}.$$

To finish, assume that A_H is a compact perturbation of a bounded Hankel operator H with matrix $(a_{n+m})_{n,m=0}^{\infty}$.

Then, the $(2^k, 2^k)$-th entry of $A_H - H$ $(k \geqslant 0)$ is $1 - a_{2^{(k+1)}}$. If $A_H - H$ was compact, the $(2^k, 2^k)$-th entry of $A_H - H$ would have to go to zero as $k \to \infty$; that is, $a_{2^{(k+1)}} \to 1$ as $k \to \infty$, but since H is bounded, we must have $a_{2^{(k+1)}} \to 0$ as $k \to \infty$. Thus we have a contradiction. □

Notice that the matrix A_H is not (strongly) asymptotically Toeplitz (not even weakly asymptotically Toeplitz). Thus not all operators in ess Hank are (strongly) asymptotically Toeplitz.

4. *Open questions*

We leave the following related questions open. Which bounded Rhaly matrices R with $R \in$ ess Hank are compact perturbations of bounded Hankel operators? Which bounded Rhaly matrices are in ess Hank and have trivial index function?

Another class of examples is the following. Consider the lower-triangular truncation of the Hilbert matrix, namely,

$$H_- = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/3 & 0 & 0 & 0 & \cdots \\ 1/3 & 1/4 & 1/5 & 0 & 0 & \cdots \\ 1/4 & 1/5 & 1/6 & 1/7 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix H_- defines a bounded operator since it is non-negative entrywise and it is less than or equal to the Hilbert matrix entrywise (see Choi [7] for a proof of the boundedness of the Hilbert matrix). A calculation shows that

$$S^{*}H_{-} - H_{-}S \in \mathscr{K},$$

so $H_{-} \in \operatorname{ess}\operatorname{Hank}$.

It can be seen (look at its real part) that H_{-} is not compact. Is H_{-} a compact perturbation of a Hankel operator? (It is easy to see that H_{-} minus the Hilbert matrix cannot be compact, but there could be another Hankel operator Γ such that $H_{-} - \Gamma$ is compact!)

It is known [**3**, **15**] that lower truncations of Hankel matrices result in bounded operators whenever the Hankel operators themselves are bounded. In fact, a trivial argument shows that these truncations are compact precisely when the original operator is compact.

All of these truncations can be seen to be in ess Hank by a quick computation. Are any of these non-compact truncations compact perturbations of bounded Hankel operators? Are their index functions trivial?

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